

Unit -04

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LANGUAGES :-

→ Symbol :- An lower case alphabet {a, b, c, ...} and binary number {0, 1} and decimal number {0, 1, 2, ... 9} are called symbol.

→ String :- A collection of symbol is called string.

A string is also called a seqⁿ or word or a sentence depending on its nature.

A string containing no elements is called an empty-string or null string and is denoted by Λ . The set of all string over a set A is denoted by A^* .

→ Length of a string :- A string over a set A consisting of n -symbols is called a string of length n . The length of a string x is denoted by $|x|$.

→ Statement "A collection of all string" is called sentences

Exa: - Ram is a good boy.

Language: - "A collection of sentence having a particular grammar is called language."

"or" A language over an alphabet A is a subset of the set A. the set of all strings over A.

Exam: - Let $A = \{a, b\}$ be an alphabet. Then the following sets are all language over the alphabet A.

- $L_1 = \{a, b, a, b, a, c, abc, cab\}$
- $L_2 = \{aba, aabaa, bbaab\}$
- $L_3 = \{a\}$
- $L_4 = \{a^i b^j : i \geq 1\}$

Operations on language.

Let L_1 and L_2 be two languages, then $L_1 \cup L_2 = \{x \in L_1 \vee x \in L_2\}$ is called "con-catenations of L_1 and L_2 ."

Exa: - Let $L_1 = \{a^i b^j\}$ and $L_2 = \{a^i b^j a^k, b^i a^j\}$. Then $L_1 \cup L_2 = \{a^i b^j, a^i b^j a^k, b^i a^j\}$ is con-catenations of L_1 and L_2 .

Grammar: - Grammar is a combination of 4-tuples (T, W, S, P) , where,

- 1) T - T is the set of terminal.
- 2) W - W is the set of non-terminal.
- 3) S - S is the starting symbol.
- 4) P - P is the set of production.

Exa: - $G = (\{S, A\}, \{a, b\}, S, \{S \rightarrow AA, A \rightarrow aa, A \rightarrow bb\})$

Here, N = set of non-terminal

$N = \{S, A\}$

T = set of terminal $T = \{a, b\}$

S = the starting symbol = S

P = set of production =

$\{S \rightarrow AA, A \rightarrow aa, A \rightarrow bb\}$

(4). P is a finite set of production.

(3) S is a special element of N called the starting symbol.

(2). N is a non-empty finite set whose elements are called non-terminals or variables T.N.

(1). T is non-empty finite set, whose element called terminals.

4-type (T, N, S, P) where G is a simply grammar or phrase-structure grammar

PHRASE STRUCTURE GRAMMERS

Thus, $L(G) = \{aaaa, aabb, bbaa, bbbb\}$.

$S \Rightarrow AA$
 $S \Rightarrow aabb$ or $S \Rightarrow aaaa$ or $S \Rightarrow bbaa$
 $S \Rightarrow bbbb$

$\Rightarrow$

$0^3 S 1^3 \Rightarrow 0^3 S 1^3$

we also have

$S \Rightarrow 0^2 S 1^2$
 $S \Rightarrow 0^3 S 1^3$

As $S \Rightarrow 0^2 S 1^2$ we have

$S \Rightarrow 0^1 S 1^1$

$G = (T = \{1, 0\}, N = \{S\}, S, P = \{S \Rightarrow 0^2 S 1^2, S \Rightarrow 0^1 S 1^1\})$

Ex:- consider the grammar G as defined below.

$\alpha = \alpha$

we can not in particular that $\alpha = \alpha_1 \Rightarrow \alpha_2 \Rightarrow \alpha_3 \Rightarrow \dots \Rightarrow \alpha_n = \beta$

$\alpha_1, \alpha_2, \dots, \alpha_n$ in such that

if there exists finite no. of strings

$\alpha = \beta$

produces β written as

string over T.N., we say that α grammar and let α and β be two

let $G = (T, N, S, P)$ be a

Derivation

Language of a grammar -

- Language generated by a grammar -

Definition - The language generated by grammar G , denoted by $L(G)$, the set of all terminal strings derived from the starting symbol S . Thus.

$$L(G) = \{w \in T^* : S \xRightarrow{*} w\}$$

The element of $L(G)$ are usually called sent.

If $S \xRightarrow{*} \alpha$, then α is called a "sentention form".

Exam (01) let $G = (\{0, 1\}, \{S\}, S, \{S \rightarrow 11S, S \rightarrow 0\})$ be a grammar. Determine $L(G)$.

Solve :- Let $\{P, N, T, S\}$ be a grammar.

Given that.

$$G = (\{0, 1\}, \{S\}, S, \{S \rightarrow 11S, S \rightarrow 0\})$$

Here,

$T = \{0, 1\}$ set of terminals.

$N = \{S\}$ set of non-terminals.

$S = S$ starting symbols.

$P = \{S \rightarrow 11S, S \rightarrow 0\}$ is the set of production.

$$\because S \rightarrow 11S$$

$$S \Rightarrow 11S$$

$$S \Rightarrow 110$$

$$110 \in L(G)$$

$$\because S \rightarrow 11S$$

$$S \Rightarrow 11S \quad (\because S \rightarrow 0)$$

$$S \Rightarrow 110$$

$$11110 \in L(G)$$

$$\because S \rightarrow 11S$$

$$S \Rightarrow 11S$$

$$S \Rightarrow 1111110$$

$$1111110 \in L(G)$$

$$L = \{0, 110, 11110, 1111110, \dots\}$$

$$L = \{1^{2n}0 : n \geq 0\}$$

Ques - $L = \{a^i b^j \mid i, j \geq 1, i \neq j\}$.

• Solve - Here,

$$T = \{a, b\}$$

$$N = \{S, A, B, C\}$$

S = starting symbol

P = p is the set of production consisting of

$$\begin{aligned} S &\rightarrow A \\ A &\rightarrow aA & S &\rightarrow C \\ A &\rightarrow aB & C &\rightarrow cb \\ B &\rightarrow aAb & C &\rightarrow Bb \\ B &\rightarrow ab \end{aligned}$$

$$\begin{aligned} S &\rightarrow A \\ S &\Rightarrow A & S &\Rightarrow A \rightarrow aA \\ S &\Rightarrow aA \\ S &\Rightarrow aaB & S &\Rightarrow A \rightarrow aB \\ S &\Rightarrow aaaBb & S &\Rightarrow B \rightarrow aBb \\ S &\Rightarrow aaaaBbb \\ S &\Rightarrow a^4 b^2 \end{aligned}$$

$$a^4 b^2 \in L_1 \cup L_2 \text{ as } i > j$$

and

$$S \rightarrow C$$

$$S \Rightarrow C$$

$$S \Rightarrow cb$$

$$S \Rightarrow Abb$$

$$S \Rightarrow aBbbb$$

$$S \Rightarrow aabbbb$$

$$a^2 b^4 \in L_2 \cup L_1 \text{ as } i < j$$

$$L_1 = \{a^i b^j \mid i > j\}$$

$$L_2 = \{a^i b^j \mid i < j\}$$

$$\therefore L = L_1 \cup L_2$$

Ques - Construct a grammar which generate at given integers upto .

Sol: - let $G = \{N, T, P, S\}$ be a grammar where,

$$N = \{S, S_1, A, B\}$$

$$T = \{0, 1, 2, \dots, 9\}$$

S = starting symbols.

P = P is the set of production consists of

$$S \rightarrow 0/2/4/6/8$$

$$S \rightarrow AS_1$$

$$A \rightarrow 1/2/3/4/5/6/7/8/9$$

$$S_1 \rightarrow 0/2/4/6/8$$

$$S \rightarrow ABS$$

$$B \rightarrow 1/2/3/4/5/6/ \dots /9$$

$$S \Rightarrow AS$$

$$S \Rightarrow (1/2/3/4/5/ \dots /9) S$$

$$S \Rightarrow (1/2/3/4/ \dots /9) \{0/2/4/6/8\}$$

$$S \Rightarrow \{10, 12, \dots, 14, 16, 18, \dots, 90, 92, 94, 96, 98\}$$

$$S \Rightarrow ABS$$

$$S \Rightarrow (1/2/3/4/ \dots /9) BS$$

$$S \Rightarrow (1/2/3/4/ \dots /9) (1/2/3/4/ \dots /9) S$$

$$S \Rightarrow (1/2/3/4/ \dots /9) \{1/2/3/4/ \dots /9\} \{0/2/4/6/8\}$$

$$S \Rightarrow (1/2/3/4/ \dots /9) \{10, 12, 14, 16, 18, \dots, 90, 92, 94, 96, 98\}$$

$$S \Rightarrow \{110, 112, 114, 116, 118, \dots, 990, 992, 994, 996, 998\} \text{ etc.}$$

Ques - Define language. The language $L(GS)$ is generated by the grammar.

$G_S = (\{S, A, B, C\}, \{a, b\}, S, \phi, \{ \})$ where ϕ is the set of production is. $S \rightarrow aS, S \rightarrow aB, B \rightarrow bc, c \rightarrow ac, c \rightarrow a$. The find the derivation for a^2ba^3 .

Soln:-

Let (P, N, T, S) be a grammar. where,

$$T = \{a, b\}$$

$$N = \{S, A, B, C\}$$

$$S = \text{starting symbol}$$

$$\phi = \phi \text{ is the set of production consists of}$$

$$S \rightarrow aS$$

$$S \rightarrow aB$$

$$S \Rightarrow aS$$

$$S \Rightarrow aaS$$

$$S \Rightarrow B \rightarrow bc$$

$$S \Rightarrow aabc$$

$$S \Rightarrow c \rightarrow ac$$

$$S \Rightarrow aabac$$

$$S \Rightarrow aabaa$$

$$S \Rightarrow c \rightarrow a$$

$$S \Rightarrow a^2ba^2$$

$$a^2ba^2 \in L(G).$$

$$\therefore S \rightarrow as$$

$$S \Rightarrow as$$

$$S \Rightarrow aab$$

$$S \Rightarrow aabc$$

$$S \Rightarrow aabac$$

$$S \Rightarrow aabdac$$

$$S \Rightarrow aabaaa$$

$$S \Rightarrow a^2ba^3$$

$$S: S \rightarrow aA^?$$

$$\{A \rightarrow bc\}.$$

$$S: c \rightarrow ac^?$$

$$S: c \rightarrow ac^?$$

$$S: c \rightarrow ac^?$$

$$a^2ba^3 \in L(G)$$

Ques $L = \{a^x b^y \mid x > y > 0\}$.

Soln - let (T, N, S, P) be a grammar where,

$$T = \{a, b\}$$

$$N = \{S, A, B\}$$

S = starting symbol

P = set of production is consist of.

$$S \rightarrow A$$

$$A \rightarrow aA$$

$$A \rightarrow aB$$

$$B \rightarrow aBb$$

$$B \rightarrow ab$$

$$S \rightarrow A$$

$$S \Rightarrow A$$

$$S \Rightarrow AB$$

$$S \Rightarrow aABb$$

$$S \Rightarrow aaabbb$$

$$S \Rightarrow a^3b^2$$

$$S: A \rightarrow aA^?$$

$$S: B \rightarrow aBb^?$$

$$S: B \rightarrow ab^?$$

$$a^3b^2 \in L(G)$$

$$S \rightarrow A$$

$$S \Rightarrow A$$

$$S \Rightarrow aA$$

$$S \Rightarrow aAB$$

$$S \Rightarrow aaabbb$$

$$S \Rightarrow aaabbb$$

$$S \Rightarrow a^4b^2$$

$$S: A \rightarrow aA^?$$

$$S: A \rightarrow aB^?$$

$$a^4b^2 \in L(G) \quad a > b > 0$$

Ques - show that the language $L(G) = \{a^n b^n c^n \mid n \geq 1\}$ is generated by the following grammar.

$$G_3 = \langle \{S, B, C\}, \{a, b, c\}, S, \phi \rangle$$

where ϕ consists of the productions.

$S \rightarrow aSBC$

$S \rightarrow abc$

$CB \rightarrow BC$

$AB \rightarrow ab$

$bB \rightarrow bb$

$bc \rightarrow bc$

$cc \rightarrow cc$

then find the derivation for the string $a^2b^2c^2$.

Soln — Let (T, N, P, S) be a grammar.
where,

$T = \{a, b, c\}$

$N = \{S, A, B, C\}$

S = string symbol.

$\phi = \phi$ is the set of production.
is consists of

(i) To derive $\rightarrow a^2b^2c^2$

$S \rightarrow aSBC$

$S \Rightarrow aSBC$

$S \Rightarrow aaSBCB$

$S \Rightarrow aaBACC$

$S \Rightarrow aabACC$

$S \Rightarrow aabhbcc$

$\therefore S \rightarrow abc$

$\therefore CB \rightarrow BC$

$\therefore AB \rightarrow ab$

$\therefore bB \rightarrow bb$

$S \Rightarrow aabhbcc$

$S \Rightarrow aabhbcc$

$S \Rightarrow a^2b^2c^2$

$\therefore bc \rightarrow bc$

$\therefore cc \rightarrow cc$

(ii) To derive $a^3b^3c^3$.

$S \rightarrow aSBC$

$S \Rightarrow aSBC$

$S \Rightarrow aaSABCB$

$S \Rightarrow aaABCBBC$

$S \Rightarrow aaabABCBCC$

$S \Rightarrow aaabABCBCC$

$S \Rightarrow aaabhbBCCC$

$\therefore S \rightarrow aSBC$

$\therefore S \rightarrow abc$

$\therefore CB \rightarrow BC$

$\therefore AB \rightarrow ab$

$\therefore bB \rightarrow bb$

$S \Rightarrow aaabhbBCCC$

$S \Rightarrow aaabhbBCCC$

$S \Rightarrow aaabhbBCCC$

$S \Rightarrow aaabhbBCCC$

$S \Rightarrow a^3b^3c^3$

$\therefore CB \rightarrow BC$

$\therefore bB \rightarrow bb$

$\therefore bc \rightarrow bc$

$\therefore cc \rightarrow cc$

Hence the language.

$L(G) = \{a^n b^n c^n : n \geq 1\}$ is generated
by given grammar.

Ques - Prove that the language :-

$L(a) = \{a^n b a^n : n \geq 1\}$
is generated by the grammar -

$G = \{S, C\}, \{a, b\}, S, \emptyset\}$
where the set of production are -

$S \rightarrow aCa$
 $C \rightarrow aCa$
 $C \rightarrow b$

Solu:- let (T, N, S, P) be a grammar.
where

$T = \{a, b\}$

$N = \{S, C\}$

S = starting symbol

P = production, $(S \rightarrow aCa, S \rightarrow aCa, C \rightarrow b)$

(i) $\because S \rightarrow aCa$

$S \Rightarrow aCa$

$S \Rightarrow aba$

$S \because C \rightarrow b$

(ii) The derivation of $a^2 b a^2$:-

$\because S \rightarrow aCa$

$S \Rightarrow aCa$

$S \Rightarrow aCaCa$

$S \Rightarrow aabaa$

$S \Rightarrow a^2 b a^2$

$\because S \rightarrow aCa$

$\because C \rightarrow b$

similarly we can show that the derivated of $a^3 b a^3, a^4 b a^4 \dots$ by the grammar.

$L(a) = \{a^n b a^n : n \geq 1\}$ is generated by the grammar.

Ques / let $L_1 = \{a^2 ab\}$ $L_2 = \{a, ab, b^2\}$. then
find -

(1) $L_1 L_2$

(2) $L_2 L_1$

(3) $L_1 \cup L_2$

(4) $L_1 \cap L_2$

Solu:- (1) $L_1 L_2 = \{a^2, ab\} \cdot \{a, ab, b^2\}$

$= \{a^3, a^3b, a^2b^2, aba, a^2b^2, ab^3\}$

(2) $L_2 L_1 = \{a, ab, b^2\} \cdot \{a^2, ab\}$

$= \{a^3, a^2b, a^3b, a^2b^2, b^2a^2, ab^3\}$

(3) $L_1 \cup L_2 = \{a^2, ab\} \cup \{a, ab, b^2\}$
 $= \{a^2, ab, a, b^2\}$

(4) $L_2 \cap L_1 = \{a, ab, b^2\} \cap \{a^2, ab\}$
 $= \{ab\}$ A_2

Ques - Let $L_1 = \{a, ab, b^2\}$ and $L_2 = \{b^2, a, ba\}$ be two language over the alphabet $I = \{a, b\}$. then find $L_1 L_2$ and L_2^2 -

Sol - Given that

$$L_1 = \{a, ab, b^2\}$$

$$L_2 = \{b^2, a, ba\}$$

Then, $L_1 L_2 = \{a, ab, b^2\} \cdot \{b^2, a, ba\}$

$$= \{ab^2, a^2, ba^2b, ab^3, a^2b, a^2b^2, b^4, ab^2, ab^3a\}$$

$L_2^2 = \{b^2, a, ba\} \cdot \{b^2, a, ba\}$

$$= \{b^4, b^2a, b^3a, ab^2, a^2, ba^2, b^3a, ba^2, b^2a\}$$

Ques - If $G = \{a, L\}$, $SS\}$, $S: \{S \Rightarrow aSL, S \Rightarrow L\}$ is a grammar find $L(G)$.

Soln:- Let $\{T, N, S, P\}$ be a grammar.

$$T = \{a, L\} \quad (\text{Set of terminal})$$

$$N = \{S\} \quad (\text{Set of non-terminal})$$

$$S = S \quad (\text{Starting symbol})$$

$$P = \{S \Rightarrow aSL, S \Rightarrow L\} \quad (\text{production})$$

$$\therefore S \rightarrow L$$

$$S \Rightarrow L$$

Thus $L \in L(G)$.

Again $S \rightarrow aSL$

$$S \Rightarrow aSL$$

$$\therefore S \rightarrow L$$

$$S \Rightarrow aLL$$

Thus, $aLL \in L(G)$

Again, $S \rightarrow aSL$ $\therefore S \rightarrow aSL$

$$S \Rightarrow aSL$$

$$S \Rightarrow a a S L L$$

$$S \Rightarrow a a L L L L \quad (\therefore S \rightarrow L)$$

Thus, $aaLLLL \in L(G)$

Hence, $L(G) = \{L, aLL, aaLL, aaaaLL, \dots\}$

$$L(G) = \{a^n, n^n, n \geq 0\}$$

Ques - Let $G = (\{E, T, F\}, \{a, +, *, \epsilon\}, E, \{b\})$ a grammar, when ϕ consist of the production
 $E \rightarrow E+T, E \rightarrow T, T \rightarrow T * F, T \rightarrow F, F \rightarrow (E), E \rightarrow a$ find derivation for the expression $a * a + a$

Solve - It is given that

T = set of terminal = $\{E, T, F\}$
 N = Set of non-terminal = $\{a, +, *, \epsilon\}$
 S = starting symbol = E
 ϕ = set of production = $\{E \rightarrow E+T, E \rightarrow T, T \rightarrow T * F, T \rightarrow F, F \rightarrow (E), F \rightarrow a\}$

Derivation for expression $a * a + a$

$E \Rightarrow E+T$
 $E \Rightarrow E+T$
 $E \Rightarrow T+T$
 $E \Rightarrow T * F + T$
 $E \Rightarrow F * F + F$
 $E \Rightarrow E + E + E$
 $E \Rightarrow a * a + a$

$[\because E \rightarrow T]$
 $[\because T \rightarrow T * F]$
 $[\because T \rightarrow F]$
 $[F \rightarrow E]$
 $[E \rightarrow a]$

Types of grammar and language

1) Zero Grammar :- A grammar G , which has no restriction on its production is called type zero-grammar. It is denoted by T_0 .

2) T_1 - A grammar G is said to be type-1 every production $\alpha \rightarrow \beta$ has the property that $L(\alpha) \leq L(\beta)$. It is denoted by T_1 .

3) Context-free grammar :- A grammar G is said to be type 2. if every production is of the form $\alpha \rightarrow \beta$ where $p \in N$, It is denoted by T_2 and is called context-free grammar.

Exam. - $G = (\{S, A\}, \{a, b\}, S, \{S \rightarrow AA, A \rightarrow a, A \rightarrow bb\})$ is a context free grammar.

$G = (\{S\}, \{a, b\}, S, \{S \rightarrow ab, S \rightarrow aa, S \rightarrow ba, S \rightarrow bb\})$ is a context free grammar.

(H) Regular grammar :- A grammar G is said to be of the type T_3 if $|d| \leq |A|$, $\alpha \in N$ and p was the form ab or a where $a \in T$ and $p \in N$. It is denoted by T_3 and is called regular grammar.

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(5) Context-sensitive Grammar :- (C.S.G.)

A phrase-structutive grammar $G = (N, T, P, S)$ is called a context-sensitive grammar, if each $(u, v) \in P$

Such that,

$u = \alpha u \beta$, $v = \alpha \gamma \beta$, when $u \in N$, $\alpha, \beta \in V^*$ and $\gamma \in V^*$

The phrases structutive language generated by a context-sensitive grammar is called a context-sensitive language.

Exa: - $G = \{S, A\}, \{a, b\}, S, \{S \rightarrow AA, A \rightarrow aa, A \rightarrow bb\}$ is a C.S.G.

(6) Regular set :- If a phrase structure grammar consists of only rules of the form $A \rightarrow \alpha \beta$ or $A \rightarrow \alpha$ then the grammar is called a regular grammar or right linear grammar. A language generated by a regular grammar is called a regular set.

Regular Expression :- Regular expression are useful for represent certain sets of string in the algebraic system. Regular expression satisfies the following rule.

□ Any terminal symbol i.e. an element of Σ and Λ, ϕ (null set) are regular expression.

□ If $\alpha \in \Sigma$ is a regular expression then we write it by α .

□ The union of two regular expression R_1 and R_2 written as $R_1 + R_2$ is also a regular expression.

□. The product of two regular expression R_1 and R_2 written as $R_1 \cdot R_2$ is also a regular expression.

□. The closure of regular expression R written as R^* is also a regular expression.

□. If R is a regular expression hence, (R) is also a regular expression.

Regular Set

Any set represent by a regular expression called "regular set" if a, b, c or then.

- (1). a denotes the set $\{a\}$
- (2). $a+b$ denotes the set $\{a\} \cup \{b\}$
- (3). $a \cdot b$ denotes the set $\{a\} \cdot \{b\}$
- (4). a^* denotes the set $\{\emptyset, a, aa, aaa, \dots\}$
- (5). (a) denotes the set $\{a\} \cup \{a^2\} \cup \{a^3\} \cup \dots$

Formula

- I_1 $\emptyset + R = R$ and $R + \emptyset = R$
- I_2 $QR = R$ and $R\emptyset = R$
- I_3 $\overline{RR} = R$ and $R^{\wedge} = R$
- I_4 $\Lambda^* = \Lambda$ and $\emptyset^* = \emptyset$
- I_5 $R + R = R$
- I_6 $R^*, R^* = R^*$
- I_7 $R \cdot R^* = R^* R$
- I_8 $(R^*)^* = R^*$
- I_9 $\Lambda + RR^* = R^*$ and $RR^* + \Lambda = R^*$
- I_{10} $(PQ)^* R = P(QR)^*$
- I_{11} $(p+q)^* = (p^* \cdot q^*)^* = (p^* + q^*)^*$
- I_{12} $(p+q)R = pR + qR$
and
 $R(p+q) = Rp + Rq$

PUMPING LEMMA

Let L be a state language accepted by a finite state machine with N states for any seq α whose length is N or larger. In the language L , α can be written as uvw . Such that v is non-empty and uvw is also in the language L for $i \geq 0$, where v^i denotes.

the concatenation of i copies of the seqⁿ v . (In other word $uv, uvw, uvw \dots$ are all in the language L)

Proof - Without loss of generality,

we may assume that the length of α is N .

Let $\alpha = a_1 a_2 a_3 \dots a_N$.

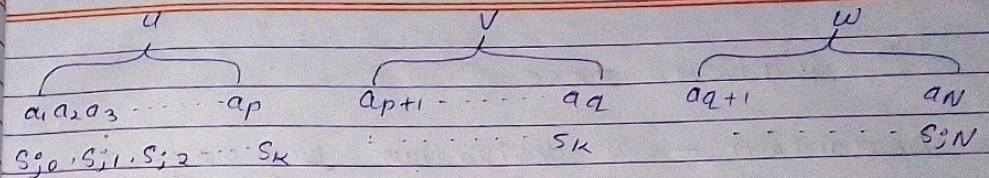
Let $s_{j_0}, s_{j_1}, \dots, s_{j_N}$ denote the states the machine visits while receiving the seqⁿ α , when

$s_{j_0} = \text{initial state}$
and $s_{j_N} = \text{accepting state}$

Since, our machine has only N states.

Therefore, among $N+1$ states $s_{j_0}, s_{j_1}, \dots, s_{j_N}$ there are two of them that are same.

Suppose, that is state s_k as shown in fig -



Where $u = a_1 a_2 a_3 \dots a_p$
 $v = a_{p+1} \dots a_q$
 $w = a_{q+1} \dots a_N$

Since $p < q \leq N$, v is non-empty and $\alpha = uvw$.

Machine starts from the initial state s_{j_0} , on applying the string u , it reaches the state s_k , on applying the string v it reaches the state s_k again.

Thus, on applying v after u the machine come back to state s_k .

It follows that after applying v^i for each $i \geq 0$, the machine is in state s_k .

On applying w at this state, machine reaches s_{j_N} , an accepting state.

Hence, $u, v^i, w \in L \quad \forall \quad i \geq 0$.

Derivation tree

Definition - A derivation tree or base tree for $CFG = (VN, \Sigma, P, S)$ is a tree satisfying the following:-

- (i) Every vertex has a label which is a variable or terminal or Λ .
- (ii) The root has label S .
- (iii) The label of an internal vertex is a variable.
- (iv) If the vertices $n_1, n_2, \dots, n_k$ exist with labels $x_1, x_2, \dots, x_k$ are the sons of vertex n with label A , then $A \rightarrow x_1 x_2 \dots x_k$ is a production in P .
- (v) A vertex n is a leaf if its label is $a \in \Sigma$ or n is the only son of its father if its label Λ .

Context-Free Grammar

G is a CFG if every production it has is of the form:

$$A \rightarrow \alpha \text{ or } A = \Lambda$$

$$A \in VN$$

$$\alpha \in (VN \cup \Sigma)^*$$

Exam:- show that the grammar described by following grammar rules is a context free grammar.

$$S \rightarrow S+S / S-S / S * S / 1 S / S / (S) / a.$$

Sol- According to production we have

$$VN = \{S\}$$

$$\Sigma = \{+, -, *, /, (,), a\}$$

Since we see that all the given productions are of the form:

$$A \rightarrow \alpha$$

where $A \in VN$ and $\alpha \in (VN \cup \Sigma)^*$, and the point to be noted here is that, in the left hand side of any production

the only variable is s which does not have any left or right context.

Therefore the given grammar is context free grammar.

• ARDEN'S Theorem:

let p and q be regular expression over Σ if p does not contain ϵ , then the following equations in R , $R = q + Rp$ has unique solution by $R = qp^*$

Proof: - To prove that,

$R = qp^*$ is a solution of the eqⁿ $R = q + Rp$

$$\begin{aligned} q + Rp &= q + (qp^*)p \\ &= q + q(p^*p) \\ &= q \cdot 1 + q(p^*p) \quad [\because q = q \cdot 1] \\ &= q(1 + p^*p) \quad [R(1+p) = Rp \Rightarrow p] \\ &= qp^* \\ &= R. \end{aligned}$$

Hence $R = qp^*$ is a solution of $R = q + Rp$

Pumping lemma for Regular sets: -

Statement: - let $M = [Q, \Sigma, \delta, q_0, F]$ be a finite automaton with states.

let L be a regular set accepted by M . let $w \in L$ and $|w| \geq m$, if $m \geq n \exists x, y$ and z such that $w = xyz$ or $w = xy^iz$ where $x, y, z \in L$ and y^i means y repeats i times.

Proof: - let $M = [Q, \Sigma, \delta, q_0, F]$ be a finite automaton.

where, Q = set of states.

Σ = set of input.

δ = transition function.

q_0 = the starting state.

F = set of final state.

let $w = a_1 a_2 a_3 \dots a_n$

then,

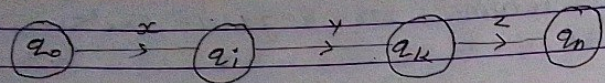
$w = a_1 a_2 a_3 \dots a_j a_{j+1} \dots a_k a_{k+1} \dots a_n$

Taking, $x = a_1 a_2 \dots a_j$

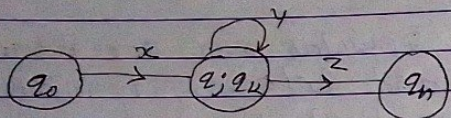
$y = a_{j+1} \dots a_k$

$z = a_{k+1} \dots a_n$

then, $w = xyz$.



If $q_i = q_k$
then the state diagram will be -



Thus, $w = xy^iz$

Hence, $w = xyz$ or $w = xy^iz$, where
 $xy, z \in L$

This called a pumping lemma.

sol b17

Exam show that $L = \{a^n b^n : n \geq 1\}$ is not regular.

Soln:- Suppose L is regular set. and
let n be no. of states in FA
accepting L .

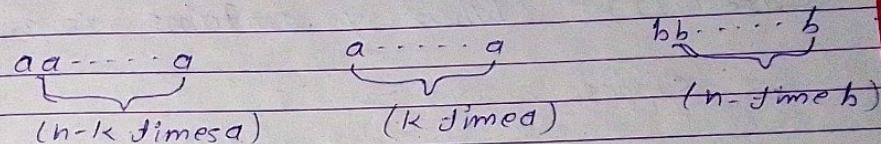
let $w = a^n b^n$ then
 $|w| = 2n > n$.

But pumping lemma, we write

$$w = xyz$$

with $|xy| \leq n, |y| \neq 0$

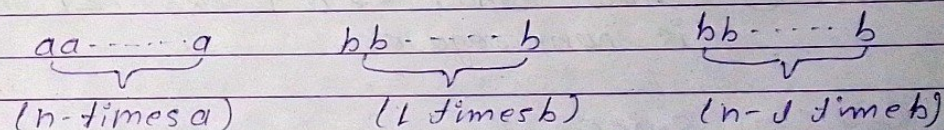
case (I). when contains only a -



but xyz is accepted by FA. Then
 xy^2z is also accepted by FA.

i.e. $a^{n-k} (a^k)^2 b^{n-k}$, i.e. $a^{n+k} b^{n-k}$ is also
accepted by FA but $a^n b^n$ is not
regular.

case (II) When y contains only b 's...



If, xyz is accepted by FA, xy^2z also
accepted by FA.

i.e. $a^n (b^l)^2 b^{n-l}$ is also accepted by FA.

anb^{n-1} is also accepted by FA. But if not of the form anb^n hence w is not regular.

Case (III) :- When y contains a 's and b 's.

Suppose, $xyz =$

$\underbrace{aa \dots a}_{(n-k \text{ times})} \underbrace{aa \dots a}_{(k \text{ times})} \underbrace{bb \dots b}_{(j \text{ times})} \underbrace{bb \dots b}_{(n-1 \text{ times})}$

but $xy^2z = a^{n-k} a^k a^j a^k b^j b^{n-1}$, which is not of the form anb^n
 $\therefore L$ is not regular.

Exam (2) Show that $\{0^i 1^m 2^m \mid i \geq 1, m \geq 1\}$ is non-regular.

Sol :- Since,

we know that the substring $1^m 2^m$ is not regular.

Hence $0^i 1^m 2^m$ is also not Regular.

Exam (4) show that $L = \{ww \mid w \in (a,b)^+\}$ is not regular.

Sol :- Let $w = anb$.
 then $L = anb^n b$.

let n be numbers of states in FA.
 accepting L by pumping lemma xy^2z .

Case (1) When y has only one b .

Now, $a^n b a^n b$. If it has two b 's
 xyz then L is not of the form ww .

hence L is regular.

Case (2) when y has only a 's let $y = a^k$

then $z = anb a^k a^{n-k} b$.

$xy^2z = anb (a^k)^2 a^{n-k} b = anb a^{n+k} b \notin L$

hence L is not regular.

✓ Kleene's theorem (2015, 16, 17).

Statements:- The class of regular sets over Σ is the smallest class containing Σ^* , for every $a \in \Sigma$ and closed under union concatenation (product) and closure.

Proof:- first we will prove the following result.

Any set L accepted by a finite automaton M is represented by a regular expression."

Let $M = (\{q_1, q_2, q_3, \dots, q_m\}, \Sigma, \delta, q, F)$ be a finite automaton.

If the string $w \in \Sigma^*$ is accepted by M , then by pumping lemma $w = xyz$, where $xy^iz \in L$ are regular expression.

Since, product of two regular expression is also a regular expression.

Therefore, $w = xy^iz$ is a regular expression.

Hence any set L accepted by a finite automaton M is represented by a regular expression.

Main proof:- Every set is a regular expression therefore, set is a regular set.

Now, if $\{a\}$ and $\{b\}$ are two regular sets.

Then $\{a\}$, $\{b\}$ are regular expression and so, $\{a\} + \{b\}$ is also a regular expression.

Thus, $\{a\} + \{b\}$ is a regular set.

Hence, unions of two regular set is also a regular set.

Again If $Sa?$ and $Sb?$ are two regular set.

Then $Sa?$ and $Sb?$ are two regular expression. and so, $Sa?.Sb?$ is also a regular expression.

Thus, $Sa?.Sb?$ is regular set.

Hence, The product of two regular set is also a regular set.

Again -

If $Sa?$ is a regular set.

Then $Sa?$ is a regular expression. and so $Sa?$ is also a regular expression.

Thus, $Sa?$ is a regular set.

Hence, the closure of regular set is a regular set.

This complete the proof.

Polish Notation.

→ Conversion of infix expression to polish notation.

This notation used between the two symbols is called in-fix notation for example pnq . This notation is called in-fix notation. Thus infix notation has the follows rule -

operand operator operand (infix)
suffix and prefix.

but the notation has the following rule -

operand operand operator (suffix)
or
operator operand operator (Prefix)

Infix

Suffix

Prefix

A

A

A

AVB

ABV

VAB

AVBVC

ABVCV

VVABC

AV(BVC)

ABCVV

VAVBC

a+b ab ab

$AV(BAC)$	$ABCV$	$VANBC$
$AN(BVC)$	$ABZVN$	$NAVBC$
$AN(BAC)$	$ABNCN$	$NNABC$
$AN7B$	$AB7A$	$NA7B$

The polish notation for expression has the rules.

(1) polish notation is a well formed formula.

2) In polish notation single symbol is called C, formula.

3) The range of a symbol x_i is 1

Infix	Rank	suffix	polish	well formed
$a+b+c$	1	$ab+c$	$+$	not well for
$ab+c$	2	$abc+$	$+$	not well for
$a+(b+c)$	1	$abc++$	$+$	not well for
$a+*b$	0	$ab+*$	$+$	not well for
$a*(b+c)$	1	$abc+*$	$+$	not well for
$(a+b)+c$	1	$abc++$	$+$	not well for
$(a+b)*(c-d)$	1	$ab+cd-*$	$+$	well for

Theorem - The rank of any well formed polish formula is 1 and the rank of any proper head of a polish formula is greater than or equal to 1. conversely if the ranks are as described then the formula is well formed.

Proof - The proof will be induction on the length of formula.

If we have a formula of length 1. then must be a symbol and it has a rank of 1.

This complete the basic step.

We know proceed the induction step. assume that every formula of length $\leq n$ has a rank 1.

consider the form $x_1 x_2 \dots x_i o_i$ with $d(x_i) \leq n = 1, 2, \dots, p$ where each x_i is well formed.

By the induction hypothesis, each $x_i o_i$ has a rank of 1.

consequently, the rank of $x_1 x_2 \dots x_p 0_i$ must be

$$(1-p) + [1+1+1+\dots+1] \\ 1-p+p=1 \\ L=1$$

(p times)

This is true and induction is complete also, if y is a proper head of $x_1 x_2 \dots x_p 0_i$.

$$\begin{aligned} \text{Then, Rank of } y &= 1-p + [1+1+\dots+1] \\ &= 1-p+p+1 \\ &= 2 \\ &> 1 \end{aligned}$$

(n times)

Hence the rank of any proper head of a polish formula is greater than 1.

Conversely -

Suppose that the rank of symbol is 1 and the rank of operation is $1-p$ where p is the order of

operation consider a formula.

$$x_1 x_2 \dots x_n 0_i$$

$$\begin{aligned} \text{Then the rank of this formula is} \\ &= (1+1+\dots+1) + (1-n) \\ &= n+1-n \\ &= 1 \end{aligned}$$

Hence, the formula is well formed.

The complete the proof.

UNIT-03

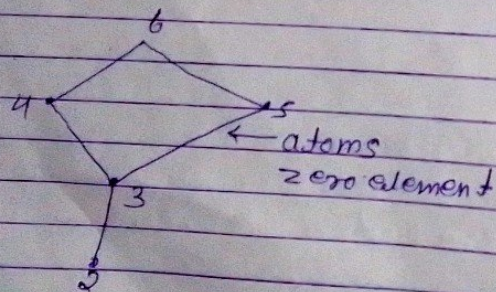
Join irreducible

Definition :- let $(L, \leq)$ be a lattice with a lower bound 0. An element $a \in L$ is said to be a join-irreducible if $a = x + y \Rightarrow a = x$ and $y = 0$

Exa :- $a \in L$ is join-irreducible.

Let $a \neq 0$ be an element of L . the element "a" is join irreducible if and only if "a" has unique immediate predecessor.

Atoms :- The join-irreducible element of lattice L , which immediately succeed 0 are called "atoms".



Minterm of minimal Boolean function.

let $(B, +, \cdot, 0, 1)$ be a boolean algebra. A boolean algebra expression of n -variable $x_1, x_2, \dots, x_n$ is said to be minterm or minimal Boolean function if it is of the form.

$$\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n$$

where $\tilde{x}_i$ denoted either x_i or x_i' .

Exa :- $x_1 * x_2 * x_3', x_1' * x_2 * x_3, x_1 * x_2 * x_3, x_1 * x_2' * x_3'$ etc are minterms of variables x_1, x_2, x_3

$$x_1 \oplus x_2' * x_3 = x, x_1 * x_3 = x$$

MAXTERM OR MAXIMAL BOOLEAN FUNCTIONS

let $(B, (+, \cdot, 1, 0))$ be a boolean algebra. A boolean algebra expression of n -variable $x_1, x_2, \dots, x_n$ is said to

be a maxterm. if it is of the form

$$\tilde{x}_1 \oplus \tilde{x}_2 \oplus \dots \oplus \tilde{x}_n$$

Where $\tilde{x}_i$ denoted either x_i or x_i' .

Exa:- $x_1 \oplus x_2 \oplus x_3'$, $x_1' \oplus x_2 \oplus x_3$
etc are maxterm of
variable x_1, x_2, x_3 .

✓ Boolean Algebra.

(0, 1, +, ·, ')

$$x + x' = 1$$

$$x \cdot x' = 0$$

$$x + 0 = x$$

$$x \cdot 1 = x$$

$$x + 1 = 1$$

$$x \cdot 0 = 0$$

Definition:- let B be a non-empty set.
with two binary operation '+'
and '·' and a binary operation and
B contains least element, denoted

by 0, and a greatest element denoted by 1.

Then (B, +, ·, ') is called a boolean algebra if it has the following property.

(1) [B1] Commutative law:-

$$a + b = b + a \quad \forall a, b \in B$$

$$a \cdot b = b \cdot a \quad \forall a, b \in B$$

(2) [B2] Distributive laws:-

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

$$a + (b \cdot c) = (a + b) \cdot (a + c)$$

(3) [B3] Identity law.

$$a + 0 = a = 0 + a$$

$$a \cdot 1 = a = 1 \cdot a \quad \forall a \in B$$

(4) [B4] Complement law.

$$a + a' = 1 = a' + a$$

$$a \cdot a' = 0 = a' \cdot a$$

Ex: 2a.

- 1) $a+a=a, a \vee a=a$
- 2) $a \cdot a=a, a \wedge a=a$
- 3) $a+a'=1 \rightarrow a=1, a'=0 \Rightarrow 1+0=1$
- 4) $a \cdot a'=0 \rightarrow a=1, a'=0 \Rightarrow 1 \cdot 0=0$

5) $(a')'=a$

6) $1+a=1 \begin{cases} 1+0=0 \\ 1+1=1 \end{cases} \text{ or } (1 \vee a=1)$

7) $1 \cdot a=a \begin{cases} 1 \cdot 0=0 \\ 1 \cdot 1=1 \end{cases}$

8) $a+bc=(a+b) \cdot (a+c)$

9) $a \cdot (b+c)=a \cdot b + a \cdot c$

Ex: 3: Let S be a non-empty set and $P(S)$ be its power set. show that the set algebra $(P(S), \cap, \cup, \phi, S)$ is a boolean algebra.

Proof: — Let $P(S)$ be a power set of S . Then $P(S)$ is a set of all subsets of S .

(I) Commutative of law.

Since, $\cap$ and $\cup$ are commutative operation. Therefore,

$$A, B \in P(S) \Rightarrow \begin{cases} A \cap B = B \cap A \\ A \cup B = B \cup A \end{cases}$$

$\Rightarrow$ This commutative law holds in $P(S)$.

(II) Distributive law: —

Let $A, B, C \in P(S)$.

Then, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Thus the distributive law holds in $P(S)$.

(III) Identity law: — Since, $A \cup \phi = A$,
 $A \cap S = A$.

$\therefore \phi$ is the least or zero element and S is the unit element.

(IV). Complement law -

Let $A' = S - A$ be a complement of A .

$$\text{Then, } A \cap A' = A \cap (S - A) = \phi$$

$$\text{and } A' \cap A = (S - A) \cap A = \phi$$

$$\text{and } A \cup A' = A \cup (S - A) = S$$

$$\text{and } A' \cup A = (S - A) \cup A = S.$$

Thus complement law holds in $P(S)$.

Hence, $(P(S); \cap, \cup, \phi, S)$ is a boolean algebra.

Theorem (Idempotent law) - If a be an element of a boolean algebra $(B, +, \cdot)$ then -

$$(i) \quad a + a = a$$

$$(ii) \quad a \cdot a = a.$$

Proof:- (i) let $a \in B$ then $a + a \in B$

$$(a + a) = (a + a) \cdot 1$$

$$= (a + a)(a + a')$$

$$= (a + aa')$$

$$= a + 0$$

$$= a$$

$$\because a + a' = 1$$

$$\because aa' = 0$$

$$\boxed{a + a = a}$$

(ii) let $a \in B$ then $a \cdot a \in B$.

$$aa = a \cdot a + 0$$

$$= aa + aa'$$

$$= a(a + a')$$

$$= a \cdot 1$$

$$= a \cdot 1$$

$$= a$$

$$\boxed{a \cdot a = a}$$

Theorem - for every element a in a boolean algebra.

$$(i) \quad a + 1 = 1$$

$$(ii) \quad a \cdot 0 = 0.$$

Proof:- (i) Consider,

$$(a + 1) = (a + 1) \cdot 1$$

$$= (a + 1)(a + a')$$

$$= a + (1a')$$

$$= a + a'$$

$$= 1$$

$$\boxed{a + 1 = 1}$$

(iii) consider,

$$\begin{aligned} a \cdot 0 &= a \cdot 0 \neq 0 \\ &= a \cdot 0 + aa' \quad [\because aa' = 0] \\ &= a(0+a') \\ &= aa' \quad [\because 0+a' = a'] \\ &= 0 \end{aligned}$$

$$\boxed{a \cdot 0 = 0}$$

Theorem (4) Absorption law:-

for each pair of element a and b in a boolean algebra B .

(i) $a + (ab) = a$

(ii) $a(a+b) = a$

Sol - (i) consider,

$$\begin{aligned} a + ab &= a \cdot 1 + a \cdot b \\ &= a(1+b) \\ &= a(b+1) \quad [\because a+1 = 1] \\ &= a \cdot 1 \\ &= a \end{aligned}$$

$$\boxed{a + ab = a}$$

(2) consider,

$$\begin{aligned} a(a+b) &= (a+0)(a+b) \\ &= (a+0 \cdot b) \\ &= a+b \cdot 0 \\ &= a+0 \\ &= a \end{aligned}$$

$$\boxed{a(a+b) = a}$$

Theorem (5) In a boolean algebra B -

(i) $0' = 1$

(ii) $1' = 0$

(iii) $(a')' = a \quad \forall a \in B$.

Proof :- consider,

(i) $0' = 0' + 0$
 $= 1$

(ii) $1' = 1' \cdot 1$
 $= 1 \cdot 1'$
 $= 0$

(iii) Using the complement axiom -

$$a + a' = 1 \quad \text{or} \quad aa' = 0 \quad \forall a \in B.$$
$$a \rightarrow a'$$

$$\therefore a' + (a')' = 1 \quad \text{and} \quad (a')(a')' = 0.$$

$$(a')' + a' = 1 \text{ and } (a')(a')' = 0$$

$$(a')' = a$$

Exam (8) In boolean algebra $(B, +, \cdot, ')$

[300] If $a \cdot x = b \cdot x$ and $a \cdot x' = b \cdot x'$ then show that $\boxed{a=b}$

Sol- The given statement are-

$$a \cdot x = b \cdot x$$

$$a \cdot x' = b \cdot x'$$

$$\text{Add } a = a' \cdot 1$$

$$= a \cdot (x + x')$$

$$= a \cdot x + a \cdot x'$$

$$= b \cdot x + b \cdot x'$$

$$= b \cdot (x + x')$$

$$= b \cdot 1$$

$$\boxed{a=b}$$

Exam (9) Show that a boolean algebra cannot have three-element.

Sol- Let $(B, +, \cdot, ')$ be a boolean algebra where $B = \{0, 1, a\}$.

Since, 0 and 1 are the identity element of B with respect to the binary compositions $+$ and $\cdot$.

Respectively, -

$$\text{Also, } 0' = 1 \text{ and } 1' = 0.$$

Also we know that complement of any elements is unique.

Therefore a' does not exist in B .

Hence $(B, +, \cdot, ')$ cannot be a Boolean algebra when B contains three element.

2016.

Prove that the following boolean identities

$$(a) \quad a \oplus (a' * b) = a \oplus b$$

$$\Rightarrow a \oplus (a' * b)$$

$$\Rightarrow (a \oplus a') * (a \oplus b)$$

$$\Rightarrow 1 * (a \oplus b)$$

$$\Rightarrow a \oplus b$$

$$(b) \quad a * (a' \oplus b) = a * b$$

$$\Rightarrow a * (a' \oplus b)$$

$$\Rightarrow (a * a') \oplus (a * b)$$

$$\Rightarrow 0 \oplus (a * b)$$

$$\Rightarrow (a * b) \oplus 0$$

$$\Rightarrow a * b$$

$$a + 1 = 1 + a$$

$$a + 0 = a$$

$$a \cdot a' = 0$$

$$(c) \quad (a * b * c) \oplus (a * b) = a * b$$

$$\Rightarrow (a * b * c) \oplus (a * b)$$

$$\Rightarrow (a * b * c) \oplus (a * b) * 1$$

$$\Rightarrow (a * b) * (c \oplus 1)$$

$$\Rightarrow (a * b) * 1$$

$$\Rightarrow a * b$$

$$(d) \quad a = b \Rightarrow a \cdot b' + a' \cdot b = 0$$

$$a = b \Leftrightarrow a \cdot b' + a' \cdot b = 0$$

$$a = b \Leftrightarrow a \cdot b' = b \cdot b' = 0$$

$$\text{and } a' \cdot b = a' \cdot a = 0$$

$$\Leftrightarrow a \cdot b' + a' \cdot b = 0 + 0$$

$$\Leftrightarrow a \cdot b' + a' \cdot b = 0$$

$$(e) \quad a = 0 \Leftrightarrow a \cdot b' + a' \cdot b = b$$

$$a = 0 \Leftrightarrow a \cdot b' + a' \cdot b = b$$

$$a = 0 \Leftrightarrow a = 0, a' = 1$$

$$\Leftrightarrow a \cdot b' = 0 \cdot b' = 0$$

$$\text{and } \Leftrightarrow a' \cdot b = 1 \cdot b = b$$

$$\Leftrightarrow a \cdot b' + a' \cdot b = 0 + b$$

$$\Leftrightarrow a \cdot b' + a' \cdot b = b$$

Ques:- show that

$$(a + b')(b + c')(c + a') = (a' + b)(b' + c)(c' + a)$$

Sol:- L.H.S.

$$\Rightarrow (a + b')(b + c')(c + a')$$

$$\Rightarrow (ab + ac' + bb' + b'c')(c + a')$$

$$\Rightarrow (ab + ac' + 0 + b'c')(c + a')$$

$$\Rightarrow (ab + ac' + b'c')(c + a')$$

$$\Rightarrow abc + abd' + ac'c + ad'c + b'c'c + b'c'a'$$

$$\Rightarrow abc + a + 0 + 0 + 0 + b'c'a'$$

$$\Rightarrow abc + a'b'c'$$

R.H.S.

$$\Rightarrow (a' + b)(b' + c)(c' + a)$$

$$\Rightarrow (a'b' + a'c + bb' + bc)(c' + a)$$

$$\Rightarrow (a'b' + a'c + 0 + bc)(c' + a)$$

$$\Rightarrow (a'b' + a'c + bc)(c' + a)$$

$$\Rightarrow a'b'c' + a'ab' + a'cc' + a'ac + bcc' + bca$$

$$\Rightarrow a'b'c' + 0 + 0 + 0 + 0 + abc$$

$$\Rightarrow abc + a'b'c'$$

$$\underline{\underline{L.H.S. = R.H.S.}}$$

Ques - In any boolean algebra show that

$$(i) (a+b)(a'+c) = ac + a'b' = ac + a'b + bc$$

$$(ii) a \leq b \Rightarrow a + bc = b(a+c)$$

Soln - (i) L.H.S. = $(a+b)(a'+c)$
 $= a \cdot a' + ac + ba' + bc$
 $= a + ac + ba' + bc$
 $= ac + a'b + bc$ R.H.S

(ii) $a \leq b \Rightarrow a \cdot b = a$
 $\Rightarrow a + bc = ab + bc$
 $\Rightarrow a + bc = ba + bc$
 $\Rightarrow a + bc = b(a+c)$

Ques :- Simplify the following boolean expression

(a) $(a * b)' \oplus (a \oplus b)'$

Soln - $\Rightarrow (a * b)' \oplus (a \oplus b)'$

$$\Rightarrow [a' \oplus b'] \oplus (a' * b')$$

$$\Rightarrow [a' \oplus b' \oplus a'] * [a' \oplus b' \oplus b']$$

$$\Rightarrow [(a' \oplus a') \oplus b'] * [a' \oplus b']$$

$$\Rightarrow a' \oplus b'$$

(b) $(a' * b' * c) \oplus (a * b' * c') \oplus (a * b' * c')$

Soln - $\Rightarrow (a' \oplus a) * (b' * c) \oplus (a * b' * c')$
 $\Rightarrow [1 * (b' * c)] \oplus (a * b' * c')$
 $\Rightarrow (b' * c) \oplus (a * b' * c')$

$$\Rightarrow (b' \oplus c) * (b' \oplus b) * (b' \oplus c') * (c \oplus a) * (c \oplus b') * (c \oplus c')$$

$$\Rightarrow (b' \oplus a) * b' * (b' \oplus c') * (c \oplus a) * (c \oplus b') * 1$$

$$\Rightarrow (b' \oplus a) * b' * (c \oplus a) * (c \oplus b')$$

$$\Rightarrow b' * (c \oplus a) * (c \oplus b')$$

$$\Rightarrow (c \oplus a) * b' * (c \oplus b')$$

$$\Rightarrow (c \oplus a) * b'$$

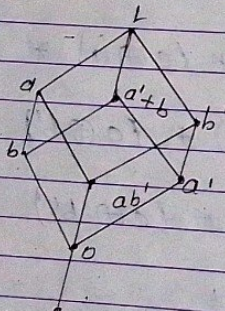
Sub - Boolean Algebra

Definition :- Let $(B, \oplus, *, ', 0, 1)$ be a boolean algebra and $S \subseteq B$.

If $(S, \oplus, *, ', 0, 1)$ is a boolean algebra.

Then S is called sub boolean algebra of B .

Exa:- Consider the boolean algebra-



The subset $S = \{a, a', 0, 1\}$ is sub-boolean algebra of B .

Homomorphism of boolean Algebra.

Let B_1 and B_2 be two boolean algebra a mapping $f: B_1 \rightarrow B_2$ is said to be homomorphism if-

- (1) $f(a+b) = f(a) + f(b)$
- (2) $f(a \cdot b) = f(a) \cdot f(b)$
- (3) $f(a') = [f(a)]'$

Boolean Isomorphism.

Let B_1 and B_2 be two boolean algebra A mapping $f: B_1 \rightarrow B_2$ is said to be boolean isomorphism if-

- (1) f is one-one
- (2) f is onto-
- (3) f is homomorphism.

Equivalent boolean Algebra.

If there is an isomorphism between two boolean algebra B_1 and B_2 then B_1 and B_2 are called equivalent boolean algebra.

Boolean expression :-

Let $(B, +, \cdot, ')$ be a boolean algebra.
A Boolean expression over $(B, +, \cdot, ')$ is called as follows -

- ① Any element of B is a boolean expression
- ② Any variable name is a boolean expression

Boolean expression of n -variables -

Let $(B, +, \cdot, ', 0, 1)$ be a boolean algebra. A boolean expression $E(x_1, x_2, \dots, x_n)$ over B that contain n distinct variables $x_1, x_2, \dots, x_n$ is usually referred to as boolean expression of n -Variable.

Exa - $E(x_1, x_2, x_3, x_4) = (x_1 * x_2) (+) (x_2 * x_3) (+) (x_1 * x_3)$

Equivalent boolean Expression

Two boolean expression E_1 and E_2 are said to be equivalent if they assume the same value for every assignment of values of variables $x_1, x_2, \dots, x_n$ we write,

$$E_1(x_1, x_2, x_3, \dots, x_n) = E_2(x_1, x_2, \dots, x_n)$$

Exam - if $E_1(x_1, x_2, x_3) = (x_1 + x_2) \cdot (x_1 + x_3)'$
 $E_2(x_1, x_2, x_3) = x_1 + (x_2 \cdot x_3)'$

Then E_1 and E_2 are equivalent.

Boolean function

A function $f: B^n \rightarrow B$ from B^n to B is called a Boolean function if it can be specified by a boolean expression of n -variables.

Boole's theorem

Statement :- The number of minimal boolean function in n -variable are 2^n .

Proof :- Suppose $x_1, x_2, \dots, x_n$ are n -variables in a Boolean algebra $(B, \oplus, \cdot, 0, 1)$ and let their complements be respectively $x_1', x_2', x_3', \dots, x_n'$.

Now there are two ways of selecting the first variable x_1 or x_1' in a minimal boolean function. There are two ways of selecting the second variable x_2 or x_2' , ... there are two ways to selecting n th variable x_n or x_n' .

Thus, we see that there are two ways to selecting each of the n -variables.

Hence, The number of minimal boolean function in n variable is $= 2 \times 2 \times 2 \dots \times 2 = 2^n$.

Disjunctive Normal form or conical form :-

Definition - A boolean expression over boolean algebra $(B, \oplus, \cdot, ', 0, 1)$ is said to be disjunctive normal form (DNF) if it is a join of minterms.

Exam:- $(x_1 \cdot x_2 \cdot x_3) \oplus (x_1' \cdot x_3 \cdot x_3') \oplus (x_1 \cdot x_2' \cdot x_3') \oplus (x_1' \cdot x_2' \cdot x_3')$

Complete Disjunctive Normal form or complete conical form

Defⁿ - If the number of distinct terms in disjunctive normal form of Boolean function of n -variable are 2^n , then it is called complete disjunctive normal form.

Exam:- $f(x_1, x_2) = x_1 \cdot x_2 + x_1' \cdot x_2 + x_1 \cdot x_2'$ is a complete disjunctive normal form for 2 variables.

Canonical form - A formula which is equivalent to a given formula and which consists product of elementary sums is called a canonical form.

There are two types of canonical form -

- (1) Conjunctive normal form.
- (2) Disjunctive normal form.

- (1) Find complete disjunctive normal form in two variable x_1, x_2 and show that its value is 1.

Sol - $f(x_1, x_2) = x_1 \cdot x_2 + x_1 \cdot x_2' + x_1' \cdot x_2 + x_1' \cdot x_2'$
 $= x_1(x_2 + x_2') + x_1'(x_2 + x_2')$
 $= x_1 \cdot 1 + x_1' \cdot 1$
 $= x_1 + x_1'$ $[\because x_2 + x_2' = 1]$
 $= 1$

- (2) Find complete disjunctive normal form in three variable and show that its value is 1.

Sol - The three variable is x_1, x_2, x_3 .

So $F(x_1, x_2, x_3) = x_1 \cdot x_2 \cdot x_3 + x_1 \cdot x_2 \cdot x_3' + x_1 \cdot x_2' \cdot x_3 + x_1' \cdot x_2 \cdot x_3 + x_1 \cdot x_2' \cdot x_3' + x_1' \cdot x_2 \cdot x_3' + x_1' \cdot x_2' \cdot x_3 + x_1' \cdot x_2' \cdot x_3'$

$$= x_1 \cdot x_2 \cdot (x_3 + x_3') + x_1 \cdot x_2' \cdot (x_3 + x_3') + x_1' \cdot x_2 \cdot (x_3 + x_3') + x_1' \cdot x_2' \cdot (x_3 + x_3')$$

$$= x_1 \cdot x_2 + x_1 \cdot x_2' + x_1' \cdot x_2 + x_1' \cdot x_2'$$

$$= x_1(x_2 + x_2') + x_1'(x_2 + x_2')$$

$$= x_1 + x_1' = 1$$

Question - Change the following function to disjunctive (sum of products) normal form of three variable x_1, x_2, x_3

'08'

Obtain sum of product of the following ^{boolean} express in an equivalent canonical form in three variable x_1, x_2, x_3 .

Sol - Given that,

$$f(x_1, x_2, x_3) = x_1 \cdot x_2' + x_3$$

$$= x_1 \cdot x_2' \cdot 1 + x_3 \cdot 1 \cdot 1 \cdot x_3$$

$$\begin{aligned}
 &= x_1 x_2' (x_3 + x_3') + (x_1 + x_1') \cdot (x_2 + x_2') x_3 \\
 &= x_1 x_2' x_3 + x_1 x_2' x_3' + x_1 x_2 x_3 + x_1' x_2 x_3 \\
 &\quad + x_1 x_2' x_3 + x_1' x_2 x_3 + x_1 x_2 x_3 + x_1' x_2' x_3 \\
 &= x_1 x_2 x_3 + x_1 x_2' x_3' + x_1 x_2 x_3 + x_1' x_2 x_3 \\
 &\quad + x_1 x_2' x_3 + x_1' x_2' x_3
 \end{aligned}$$

Thus, is the required sum of product canonical form.

Ques Disjunctive normal form.

$$\begin{aligned}
 (1) \quad x \cdot y &= (x \cdot y) \cdot 1 \\
 &= (x \cdot y) \cdot (z + z') \quad (\because z + z' = 1) \\
 &= x y z + x y z'
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad x + x' \cdot y &= x \cdot 1 + x' \cdot y \\
 &= x (y + y') + x' \cdot y \\
 &= x y + x y' + x' y
 \end{aligned}$$

$$(3) \quad f(x, y, z) = x \cdot y' + x \cdot z + x y$$

$$\begin{aligned}
 \text{Sol -} \quad &x \cdot y' + x \cdot z + x y \\
 \Rightarrow &x \cdot y' \cdot 1 + x \cdot 1 \cdot z + x \cdot y \cdot 1
 \end{aligned}$$

$$\Rightarrow x \cdot y' \cdot (z + z') + x \cdot (y + y') \cdot z + x \cdot y \cdot (z + z')$$

$$\Rightarrow x y' z + x y' z' + x y z + x y' z + x y z + x y z'$$

$$\Rightarrow x y z + x y' z + x y z' + x y' z'$$

$$(11) \quad (x \cdot y' + x \cdot z)' + x'$$

$$\begin{aligned}
 \text{Sol} \Rightarrow &(x y')' \cdot (x z)' + x' \\
 \Rightarrow &[x' + (y')] \cdot [x' + z'] + x' \\
 \Rightarrow &(x' + y') (x' + z') + x' \\
 \Rightarrow &(x' + y z') + x'
 \end{aligned}$$

$$\Rightarrow (x' + x') + y z'$$

$$\Rightarrow x' + y z'$$

$$\Rightarrow x' (y + y') (z + z') + (x + x') \cdot y \cdot z'$$

$$\Rightarrow (x' y + x' y') (z + z') + x y z' + x' y z'$$

$$\begin{aligned}
 \Rightarrow &x' y z + x' y z' + x y' z + x y' z' + x y z' \\
 &+ x' y z'
 \end{aligned}$$

$$\Rightarrow x' y z + x y' z + x y z' + x' y z' + x y' z'$$

⑤ $(x+y+z) \cdot (xy+x' \cdot z)'$

Sol - $(x+y+z) \cdot (xy+x' \cdot z)'$

$$\Rightarrow (x+y+z) [(x'+y') \cdot (x'z)']$$

$$\Rightarrow (x+y+z) [(x'+y') \cdot (x+z)']$$

$$\Rightarrow (x+y+z) [xx' + x'z' + y'x + y'z']$$

$$\Rightarrow (x+y+z) [0 + x'z' + y'x + y'z']$$

$$\Rightarrow xx'z' + xy'x + xy'z' + yx'z' +$$

$$xy'y + yy'z' + x'zz' + x'y'z + yy'z$$

$$\Rightarrow 0 + xy'x + x'y'z' + xy'z$$

$$\Rightarrow 0 + xy'x + xy'z' + x'y'z' + 0 + 0 + 0$$

$$+ xy'z + 0$$

$$\Rightarrow xy'x + xy'z' + x'y'z' + xy'z$$

$$\Rightarrow xy' \cdot 1 + xy'z' + x'y'z' + xy'z$$

$$\Rightarrow xy' (z+z') + x'y'z' + xy'z$$

$$\Rightarrow xy'z + xy'z' + x'y'z' + xy'z$$

$$\Rightarrow xy'z + xy'z' + x'y'z' \quad \text{Ans}$$

Conjunctive normal form:-

① $F(x,y,z) = xy' + xz + xy$

$$= x(y' + z + y)$$

$$= x(y + y' + z)$$

$$= x(1 + z)$$

$$= x \cdot z \quad \text{Ans}$$

Disjunctive:- sum of product (i) Product of sum (+) product of sum - sum of product (+)

Ques:- obtain sum of product of the following boolean expression in an equivalent canonical form in three variables x_1, x_2 and x_3 :-

Sol:- Sum of product:-

$$x_1 \cdot x_2' + x_3 = (x_1 * x_2') \oplus x_3$$

$$= [(x_1 * x_2' * (x_2 + x_3')) \oplus [(x_1 \oplus x_2') * (x_2 \oplus x_2') \oplus x_3]$$

$$= [(x_1 * x_2' * x_3) \oplus (x_1 * x_2' * x_3')] \oplus$$

$$[(x_1 * x_2 * x_3) \oplus (x_1 * x_2' * x_3) \oplus (x_1 * x_2 * x_3') \oplus (x_1 * x_2' * x_3')] \oplus [(x_1' * x_2 * x_3) \oplus (x_1' * x_2' * x_3) \oplus (x_1' * x_2 * x_3') \oplus (x_1' * x_2' * x_3')]$$

$$= (x_1 * x_2 * x_3) \oplus (x_1 * x_2' * x_3) \oplus (x_1 * x_2 * x_3') \oplus (x_1' * x_2 * x_3) \oplus (x_1' * x_2' * x_3) \oplus (x_1' * x_2 * x_3') \oplus (x_1' * x_2' * x_3')$$

$$= \text{min}_7 \oplus \text{min}_5 \oplus \text{min}_6 \oplus \text{min}_1 \oplus \text{min}_4$$

$$= \Sigma \oplus (1, 3, 4, 5, 7)$$

Product of sum:-

$$x_1 x_2' + x_3 = (x_1 \oplus x_2) * (x_2' \oplus x_3) \\ = [x_1 \oplus x_2] * [x_2 \oplus x_3] \\ = [x_1 \oplus x_2] * [x_2 \oplus x_3]$$

$$= [(x_1 \oplus x_2) * (x_2 \oplus x_3)] * [(x_1 \oplus x_2) * (x_2 \oplus x_3)]$$

$$= [(x_1 \oplus x_2) * (x_2 \oplus x_3)] * [(x_1 \oplus x_2) * (x_2 \oplus x_3)]$$

$$= \text{max}_0 * \text{max}_1 * \text{max}_2$$

$$= \Sigma * 0, 1, 2 \text{ Ans}$$

Ques- Write the following boolean expression in an equivalent sum of product form in three variable x_1, x_2 and x_3 .

$$(1) x_1 * x_2$$

$$\text{Soln} - x_1 * x_2 = x_1 * x_2 * (x_3 \oplus x_3') \\ = (x_1 * x_2 * x_3) \oplus (x_1 * x_2 * x_3') \\ = \text{min}_7 \oplus \text{min}_6 \\ = \Sigma \oplus (6, 7) \text{ Ans}$$

$$(2) x_1 \oplus x_2$$

$$\text{Soln} - x_1 \oplus x_2 = [(x_1 * x_2) \oplus (x_1' * x_2')] \oplus [(x_1 * x_2') \oplus (x_1' * x_2)]$$

$$= [(x_1 * x_2) \oplus (x_1' * x_2')] \oplus [(x_1 * x_2') \oplus (x_1' * x_2)]$$

$$= (x_1 * x_2) \oplus (x_1' * x_2') \oplus (x_1 * x_2') \oplus (x_1' * x_2)$$

$$= [(x_1 * x_2) * (x_3 \oplus x_3')] \oplus [(x_1' * x_2') * (x_3 \oplus x_3')] \oplus [(x_1 * x_2') * (x_3 \oplus x_3')] \oplus [(x_1' * x_2) * (x_3 \oplus x_3')]$$

$$\Rightarrow (x_1 * x_2 * x_3) \oplus (x_1 * x_2 * x_3') \oplus (x_1' * x_2' * x_3) \oplus (x_1' * x_2' * x_3') \oplus (x_1 * x_2' * x_3) \oplus (x_1' * x_2 * x_3) \oplus (x_1 * x_2 * x_3) \oplus (x_1' * x_2 * x_3')$$

$$= \min_7 \oplus \min_6 \oplus \min_5 \oplus \min_4 \oplus \min_3 \oplus \min_2$$

$$= 5 \oplus 2, 3, 4, 5, 6, 7 \quad \text{Ans}$$

$$(iii) (x_1 \oplus x_2)' * x_3$$

$$\text{Soln} - (x_1 \oplus x_2)' * x_3 = (x_1' * x_2') * x_3$$

$$= \min_1 \quad \text{Ans}$$

equivalent proof that:-

$$\text{Ques} - (i) (x \oplus y) * (x' \oplus z) * (y \oplus z) = a$$

$$(ii) (x * z) \oplus (x' * y) \oplus (y * z) = b$$

$$(iii) (x \oplus y) * (x' \oplus z) = c$$

$$(iv) (x * z) \oplus (x' * y) = d$$

Soln:- Alternatively as follows

$$c = (x \oplus y) * (x' \oplus z) \\ = (x \oplus y * x') \oplus (x \oplus y * z)$$

$$= (x * x') \oplus (y * x') \oplus (x * z) \oplus (y * z)$$

$$= (x' * y) \oplus (x * z) \oplus (y * z)$$

$$c = b$$

$$d = (x * z) \oplus (x' * y)$$

$$= [(x * z) \oplus x'] * [(x * z) \oplus y]$$

$$= [(x \oplus x') * (z \oplus x')] * [(x \oplus y) * (z \oplus y)']$$

$$= (x' \oplus z) * (x \oplus y) * (z \oplus y)$$

$$= (x \oplus y) * (x' \oplus z) * (y \oplus z)$$

$$d = a$$

Finally we show that

$$a = (x \oplus y) * (x' \oplus z) * (y \oplus z)$$

$$= (x \oplus y) * [(x' * y) \oplus z]$$

$$= [(x \oplus y) * (x' * y)] \oplus [(x \oplus y) * z]$$

$$= [(x \oplus y) * (y * x')] \oplus [(x \oplus y) * z]$$

$$= [(x \oplus y) * y] \oplus x' \oplus [(x * z) \oplus (y * z)]$$

$$= (y * x') \oplus (x * z) \oplus (y * z)$$

$$\boxed{a = b} \quad \text{and}$$

$$c = (x \oplus y) * (x' \oplus z)$$

$$= (x * x') \oplus (x * z) \oplus (y * x') \oplus (y * z)$$

$$= 1 \oplus (x * z) \oplus (x' * y) \oplus (y * z)$$

$$= (x * z) \oplus (x' * y) \oplus (y * z)$$

$$= (x * z) \oplus (x' * y \oplus y) * z$$

$$= (x * z) \oplus (y * z)$$

$$= [(x * z) \oplus z] * y$$

$$= [(z * y) * 1]$$

$$= (z * y) * (x + x')$$

$$= (x * z) \oplus (x' * y) = d$$

$$\boxed{c = d}$$

Ques - show that the following Boolean expression Boolean algebra (B, *, ' ⊕, 0, 1) are equivalent are another -

$$(a) (x \oplus y) * (x' \oplus z) * (y \oplus z)$$

$$(b) (x * z) \oplus (x' * y) \oplus (y * z)$$

$$(c) (x \oplus y) * (x' \oplus z)$$

$$(d) (x * z) \oplus (x' * y)$$

Solⁿ - By the truth table.

$$a = (x \oplus y) * (x' \oplus z) * (y \oplus z)$$

$$b = (x * z) \oplus (x' * y) \oplus (y * z)$$

$$c = (x \oplus y) * (x' \oplus z)$$

$$d = (x * z) \oplus (x' * y)$$

x	y	z	x'	y'	z'	x+y	x'+z	y+z	x*z	x'*y	y*z
0	0	0	1	1	1	0	1	0	0	0	0
0	0	1	1	1	0	0	1	1	0	0	0
0	1	0	1	0	1	1	1	1	0	1	0
0	1	1	1	0	0	1	1	1	0	1	1
1	0	0	0	1	1	1	0	0	0	0	0
1	0	1	0	1	0	1	1	1	1	0	0
1	1	0	0	0	1	1	0	1	0	0	0
1	1	1	0	0	0	1	1	1	1	0	1

a	b	c	d
0	0	0	0
0	0	0	0
1	1	1	1
1	1	1	1
0	0	0	0
1	1	1	1
0	0	0	0
1	1	1	1

$$\therefore a=b=c=d$$

Thus, The boolean expression is equivalent

2012

Ques (2) Show that the Boolean expressions are equivalent.

1) $(x \oplus y) * (x' \oplus z) * (y \oplus z)$

ii) $(x * z) \oplus (x' * y) \oplus (y * z)$

3) $(x \oplus y) * (x' \oplus z)$

Solⁿ - By the truth table

(i)

x	y	z	x'	x+y	x'+z	y+z	(i)
0	0	0	1	0	1	0	0
0	0	1	1	0	1	1	0
0	1	0	1	1	1	1	1
0	1	1	1	1	1	1	1

1	0	0	0	1	0	0	0
1	0	1	0	1	1	1	1
1	1	0	0	1	0	1	0
1	1	1	0	1	1	1	1

(ii)	$x * z$	$x' * y$	$y + z$	(ii)
	0	0	0	0
	0	0	0	0
	0	1	0	1
	0	1	1	1
	0	0	0	0
	1	0	0	1
	0	0	0	0
	1	0	1	1

(iii)

$a \oplus y$	$x' \oplus z$	(iii)
0	1	0
0	1	0
1	1	1
1	1	1
1	0	0
1	1	1
1	0	0
1	1	1

$$(i) = (ii) = (iii)$$

The boolean expression are equivalent.

2014

Ques - Let $B = \{0, 1\}$ be a set the operation $*$, $\oplus$ and on B are given by the follows table.

	$*$	0	1	$\oplus$	0	1	$\times$	$\times$
0	0	0	0	0	0	1	0	1
1	1	0	1	1	1	1	1	0

Show that the algebra $\langle B, \oplus, *, ', 0, 1 \rangle$ is only boolean algebra.

Solⁿ - It is clear that the table $*$, $\oplus$, and $'$ are binary operations and that is binary operations on B .

We show that all axioms for a boolean algebra are satisfies.

1) Commutative law.

Since each row is its corresponding equal its column then in the composition table of $\oplus$ and $*$.

Therefore the commutative property is satisfying.

2) Identity law: - from the table, we see that

$$a \oplus 0 = a$$

$$\text{and } a * 1 = a \quad \forall a \in B.$$

Thus, 0 is the identity for $\oplus$ and 1 is the identity for $*$.

3) Distributive law: -

$$a \oplus (b * c) = (a \oplus b) * (a \oplus c).$$

$$\text{and } a * (b \oplus c) = (a * b) \oplus (a * c)$$

$$\forall a, b, c \in B.$$

4) Complement law: -

for given $0 \in B \exists 1 \in B$.

Such that,

$$0 + 1 = 1 \text{ and } 1 * 0 = 0$$

hence, for every a in B , $\exists a'$ in B .

Such that,

$$a + a' = 1 \text{ and } a * a' = 0$$

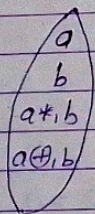
Thus, $\langle B, *, \oplus, ', 0, 1 \rangle$ is a boolean algebra.

Ques - show that a lattice homomorphism on a boolean algebra which preserves 0 and 1 is a boolean homomorphism.

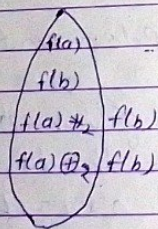
Solⁿ - lattice homomorphism :-
let $(L_1, *, \oplus_1)$ and $(L_2, *_2, \oplus_2)$ be two lattices and $f: L_1 \rightarrow L_2$ be a mapping f is said to be lattice homomorphism.

$$\text{If, } f(a *_1 b) = f(a) *_2 f(b) \\ f(a \oplus_1 b) = f(a) \oplus_2 f(b)$$

$(L_1, *_1, \oplus_1)$



$(L_2, *_2, \oplus_2)$

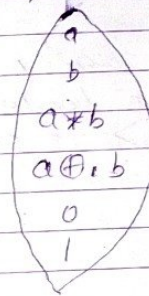


Boolean Homomorphism :- Let $(B_1, *_1, \oplus_1, 0, 1)$ and $(B_2, *_2, \oplus_2, 0, 1)$ be two boolean algebras :-
and $f: B_1 \rightarrow B_2$ be a mapping.

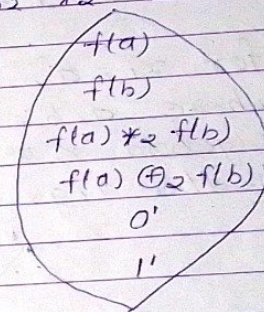
Then f is said to be boolean homomorphism.

- (1) $f(a *_1 b) = f(a) *_2 f(b)$
- (2) $f(a \oplus_1 b) = f(a) \oplus_2 f(b)$
- (3) $f(0) = 0'$
- (4) $f(1) = 1'$

$(B_1, *_1, \oplus_1, 0, 1)$



$(B_2, *_2, \oplus_2, 0, 1)$



Proof :- Suppose that $(L_1, *_1, \oplus_1)$ and $(L_2, *_2, \oplus_2)$ be two lattices f and $f: L_1 \rightarrow L_2$ is a lattice homomorphism which preserve 0 and 1.

$$\text{i.e. } f(0) = 0 \text{ and } f(1) = 1$$

Thus f satisfies the following property.

- (i) $f(a *_1 b) = f(a) *_2 f(b)$
- (ii) $f(a \oplus_1 b) = f(a) \oplus_2 f(b)$

(3). $f(0) = 0$

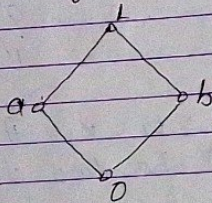
(4). $f(1) = 1$

hence f is a boolean algebra.

2011, 18.

Ques- Find the value of $x_1 * x_2 * [(x_1 * x_4) \oplus (x_3') \oplus (x_3 * x_4)']$

for $x_1 = a, x_2 = 1, x_3 = b, x_4 = 1$
where $a, b \in B$ and the boolean algebra $\langle B, *, \oplus, 1, 0, ' \rangle$ is shown in the fig. given below:-



Soln - By substituting the required elements in the expression, we get,

$$\begin{aligned} & a * 1 * [(a * 1) \oplus 1' \oplus (b * a')] \\ &= a * [a \oplus 0 \oplus (b * b')] \because (a' = b) \\ &= a * [a \oplus b] \\ &= a * [a \oplus a'] \because (b = a') \end{aligned}$$

$$\begin{aligned} &= a * 1 \\ &= a \end{aligned}$$

$$\begin{aligned} & [a' = b] \\ & [a * a' = 0] \\ & [a + a'] = 1 \end{aligned}$$

Alternatively, the given expression can be written in an equivalent form as,

$$\begin{aligned} & (x_1 * x_2 * x_1 * x_4) \oplus (x_1 * x_2 * x_2') \oplus \\ & (x_1 * x_2 * x_3 * x_4') \\ &= (x_1 * x_2 * x_4) \oplus 0 \oplus 0 \\ &= (x_1 + x_2 * x_4) \end{aligned}$$

also replacing the variable, we get

$$a * 1 * 1 = a \quad \text{proved}$$

p-243

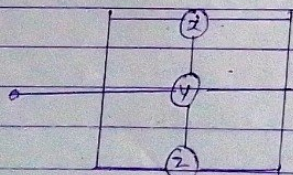
Switching circuits.

The switching is device used to control the current in an electric circuits.

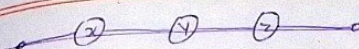
When the switch is in the "off" position we say that there is an open circuit between the terminal 1 and 2.

* Equivalent circuits :- Two switching circuits consisting of switches x, y, z are called equivalent circuits if in all position their closure are same. If S_1 and S_2 are two equivalent paths then we write $S_1 \sim S_2$.

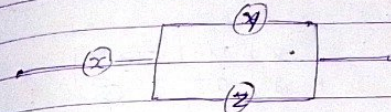
* Switching circuits :-



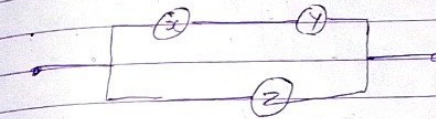
$$x + y + z$$



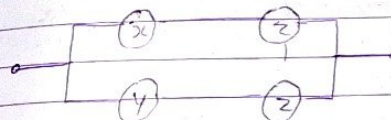
$$x \cdot y \cdot z$$



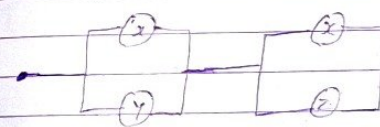
$$x \cdot (y + z)$$



$$x + y \cdot z$$

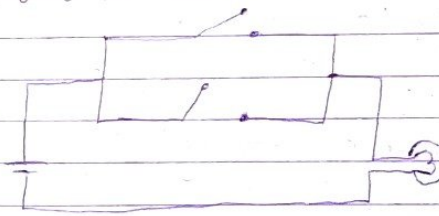


$$xz + yz$$



$$(x + y) \cdot (z + z)$$

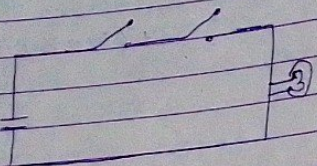
Parallel circuit :-



OR

x	y	$x + y$
0	0	0
0	1	1
1	0	1
1	1	1

Series circuits-



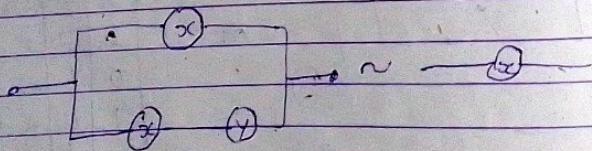
x	y	$x \cdot y$
0	0	0
0	1	0
1	0	0
1	1	1

Exa ③ for the following switching function draw a simple switching circuits.

- (i) $F(x, y) = x + x \cdot y$
 (ii) $F(x, y) = x \cdot y' + x \cdot y$

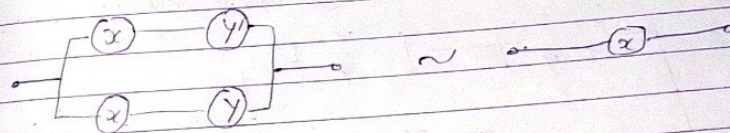
Soln - (i) $F(x, y) = x + x \cdot y$
 $= x(1 + y)$
 $= x \cdot (y + 1)$
 $= x \cdot 1$
 $= x$

$\therefore$ switching circuits are -



(ii) $F(x, y) = x \cdot y' + x \cdot y$
 $= x(y' + y)$
 $= x \cdot 1$
 $= x$

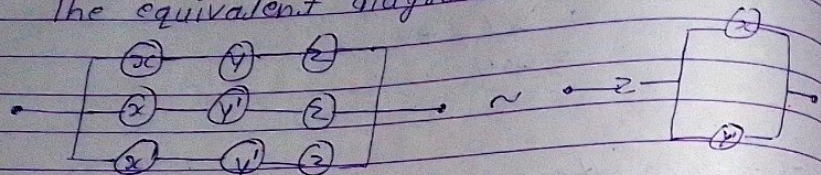
Therefore the switching circuits of $x \cdot y' + x \cdot y$ is equivalent of the switching circuits of the figure is as follows -



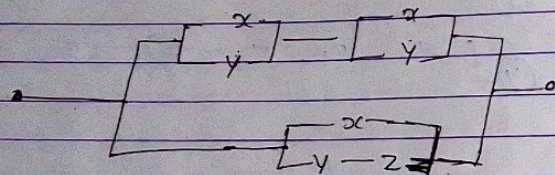
Exam:- Replace the switching from -
 $F(x, y, z) = x \cdot y \cdot z + x \cdot y' \cdot z + x' \cdot y' \cdot z$
 by a simple switching below -

Soln - $F(x, y, z) = x \cdot y \cdot z + x \cdot y' \cdot z + x' \cdot y' \cdot z$
 $= [x \cdot y + x \cdot y' + x' \cdot y'] z$
 $= [x \cdot 1 + x' \cdot y'] z$
 $= z[x + x' \cdot y']$
 $= z[1 \cdot (x + y')]$
 $= z \cdot (x + y')$

The equivalent diagram is as follows-



2017
Exam-(7) Replacing the following switching circuits by a simple one.



Soln- The switching function of the given switching circuits is given by:-

$$F(x, y, z) = (x+y) \cdot (x+z) + z \cdot (x+y \cdot z)$$

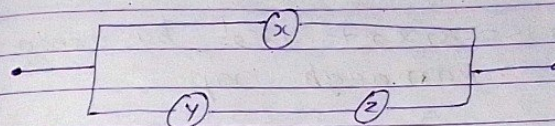
$$= (x+y \cdot z) + z(x+y \cdot z)$$

$$= (x+y \cdot z) + (1+z)$$

$$= (x+y \cdot z)$$

$$F(x, y, z) = x + y \cdot z$$

The equivalent switching circuits:-



Karnaugh map.

A Karnaugh map (or k-map) is a graphical method for representing the truth table of a boolean function.

2017
Karnaugh-map for two-variable

The Karnaugh map corresponding to the boolean expression $F(x_1, x_2)$ with two variable x_1 and x_2 is shown in the following table in which the two variable x_1 and x_2 and their negatives are combined in four way by cross reference to rows and column.

	x_1'	x_1	x_2	0	1
x_2'	$x_1'x_2'$	x_1x_2'	0	00	10
x_2	$x_1'x_2$	x_1x_2	1	01	11

2017
Exam (03) Simplify the boolean expression

$$F(x_1, x_2) = x_1 x_2 + x_1 x_2' \text{ by using karnaugh-map.}$$

Soln - The given boolean expression is

$$F(x_1, x_2) = x_1 x_2 + x_1 x_2'$$

	x_1	x_2
x_1	$x_1 x_1$	$x_1 x_2$
x_2	$x_2 x_1$	$x_2 x_2$

Thus for $F(x_1, x_2) = x_1 x_2 + x_1 x_2'$ the karnaugh map is

	x_1	x_2
x_1'	0	1
x_2	0	1

$$\text{Thus } F(x_1, x_2) = x_1 \quad \underline{A_2}$$

Exam - Simplify the boolean operator expression.

$$F(x_1, x_2) = x_1 x_2 + x_1' x_2' + x_1' x_2$$

Soln - The given below expression.

$$F(x_1, x_2) = x_1 x_2 + x_1' x_2' + x_1' x_2$$

	x_1'	x_1
x_2'	$x_1' x_2'$	$x_1 x_2'$
x_2	$x_1' x_2$	$x_1 x_2$

Thus for

$$F(x_1, x_2) = x_1 x_2 + x_1' x_2' + x_1' x_2$$

the karnaugh-map is

	x_1'	x_1
x_2'	1	0
x_2	0	1

Algebraically this can also be verified as -

$$\begin{aligned} F(x_1, x_2) &= x_1 x_2 + x_1' x_2' + x_1' x_2 \\ &= x_1 x_2 + x_1' (x_2 + x_2') + x_1' x_2 \\ &= x_1 x_2 + x_1' + x_1' x_2 \\ &= x_2 (x_1 + x_1') + x_1' \\ &\Rightarrow x_1' + x_2 \end{aligned}$$

The three variable - maps

The k-map corresponding to boolean expression $E(x_1, x_2, x_3)$ for three variable x_1, x_2, x_3 is shown in following table.

$x_1 x_2$	$x_1' x_2'$	$x_1' x_2$	$x_1 x_2'$	$x_1 x_2$
x_3	$x_1' x_2' x_3'$	$x_1' x_2 x_3'$	$x_1 x_2' x_3'$	$x_1 x_2 x_3'$
x_3'	$x_1' x_2' x_3$	$x_1' x_2 x_3$	$x_1 x_2' x_3$	$x_1 x_2 x_3$
x_3	$x_1' x_2' x_3$	$x_1' x_2 x_3$	$x_1 x_2' x_3$	$x_1 x_2 x_3$

$x_1 x_2$	00	01	11	10
0	000	010	110	100
1	001	011	111	101

$$E(x_1, x_2, x_3) = x_2' x_1' + x_2 x_3' + x_1 x_2 x_3$$

0	0	0	→	1
0	0	1	→	0
0	1	0	→	1
0	1	1	→	0
1	0	0	→	0
1	0	1	→	1
1	1	0	→	1
1	1	1	→	0

Exa (6) Find the simplified boolean expression for the function whose karnaugh map is -

1	1	1	1
1	0	0	1

Solⁿ -

	$x_1' x_2'$	$x_1' x_2$	$x_1 x_2$	$x_1 x_2'$
x_3'	1	1	1	1
x_3	1	0	0	1

$$\begin{aligned} E(x_1, x_2, x_3) &= x_1' x_2' + x_2 x_3' + x_1 x_2 x_3' \\ &= x_2' (x_1' + x_1) + x_2 x_3' \\ &= x_2' + x_2 x_3' \quad \text{Ans} \end{aligned}$$

Q. Minimize the function 2017

$$f(x, y, z, w) = xyzw + xyzw' + x'y'zw' + x'y'zw$$

Solⁿ -

xy	$x'y'$	$x'y$	xy	xy'
$z'w'$	0	0	0	0
$z'w$	0	0	0	0
zw	1	0	1	0
zw'	1	0	1	1

$$f(x, y, z, w) = x'y'z + xy + xzw' \quad \text{Ans}$$

Q. $f(a, b, c, d) = \sum (6, 7, 9, 10, 15)$

Solⁿ -

	a	b	c	d
6	0	1	1	0
7	0	1	1	1
9	1	0	0	1
10	1	0	1	0
15	1	1	1	1

Solⁿ -

c \ d	a'b'	a'b	ab	ab'
d'	a'b'c'd'	a'b'cd'	ab'cd'	ab'cd'
d	a'b'cd	a'bcd	ab'cd	ab'cd
c	a'b'cd	a'bcd	ab'cd	ab'cd
c'	a'b'cd'	a'b'cd'	ab'cd'	ab'cd'

Solⁿ -

c \ d	a'b'	a'b	ab	ab'
d'	0	0	0	0
d	0	0	0	1
c	0	1	1	0
c'	0	1	0	1

$$F(a, b, c, d) = a'bc + bcd + ab'cd + ab'cd'$$

2012

Ques - Define Boolean function what are the methods for minimization of boolean function.

Explain karnaugh-map method simplify the boolean function.

$$f(x, y, z) = x'y'z + x'yz' + xyz' + xy'z$$

using karnaugh map-method.

Solⁿ - The karnaugh map for three-variable is given by:-

	x'y'	x'y	xy	xy'
z'	x'y'z'	x'yz'	xyz'	xy'z'
z	x'y'z	x'yz	xyz	xy'z

Thus for $f(x, y, z) = x'y'z' + x'yz' + xyz' + xy'z$

The karnaugh map is -

	x'y'	x'y	xy	xy'
z'	1	1	1	0
z	0	0	0	1

$$f(x, y, z) = x'z' + yz' + xyz' \quad \text{Ans}$$

Ques (b). Explain all 5-gates used in design and implementation of digital network. The logic circuit with input xyz and output f corresponding to the boolean expression.

$$f(x, y, z) = xy'z + xyz' + xy'z'$$

Soln - $f(x, y, z) = xy'z + xyz' + xy'z'$
 $= xy'z + xyz' + xy'z'$

$$= xy'z + (y + y')xz'$$

$$= xy'z + xz'$$

$$= x[y'z + z']$$

$$= x[(y' + z')(z + z')]$$

$$= x(y' + z')$$

$$= xy' + xz'$$

	$x'y'$	$x'y$	xy	xy'
z'	0	0	1	1
z	0	0	0	1

$$F = xy' + xz'$$

Ques - Use the Karnaugh-map representation to find a minimal sum of products expression of the following function:-

$$f(a, b, c, d) = \Sigma(0, 1, 4, 5, 9, 11)$$

Soln -

	$a'b'$	$a'b$	ab	ab'
$a'd'$	1	1	0	0
$c'd$	1	1	0	1
cd	0	0	0	1
cd'	0	0	0	0

$0 \rightarrow 0 \quad 0 \quad 0 \quad 0 = a'b'c'd'$
 $1 \rightarrow 0 \quad 0 \quad 0 \quad 1 = a'b'c'd$
 $4 \rightarrow 0 \quad 1 \quad 0 \quad 0 = a'bc'd'$
 $5 \rightarrow 0 \quad 1 \quad 0 \quad 1 = a'bc'd$
 $9 \rightarrow 1 \quad 0 \quad 0 \quad 1 = ab'c'd$
 $11 \rightarrow 1 \quad 0 \quad 1 \quad 1 = ab'cd$

$$f(a, b, c, d) = a'c' + ab'd$$

$$\left\{ \begin{array}{l} a'c'd' + a'c'd + ab'd \\ a'c'(d' + d) + ab'd \\ a'c' + ab'd \end{array} \right\}$$

UNIT-02

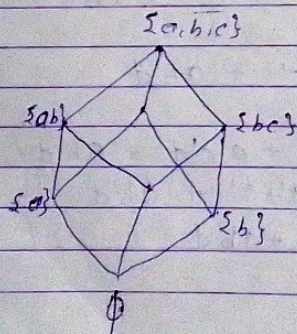
Lattice :- A lattice is a partially ordered set $\langle L, \leq \rangle$ in which every pair of element $a, b \in L$ has a greatest lower bound and a least upper bound.

The greatest lower bound of a subset $\{a, b\} \subseteq L$ will be denoted by $a * b$ and the least upper bound by $a \oplus b$.

Exam :- $A = \{a, b, c\}$.

$$P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$$

have, $(P(A), \leq)$ is a lattice
 $x, y \in P(A) \Rightarrow x \vee y \in P(A)$
 $x, y \in P(A) \Rightarrow x \wedge y \in P(A)$.



Some properties of Lattice

We shall first list some of the properties of the two binary operation of meet and join denoted by $*$ and $\oplus$ on a lattice (L) for any $a, b, c \in L$

we have.

$$(1-1) \ a * a = a$$

$$(1-2) \ a * b = b * a$$

$$(1-3) \ (a * b) * c = a * (b * c)$$

$$(1-4) \ a * (a \oplus b) = a$$

$$(1-1)' \ a \oplus a = a \text{ Idempotent}$$

$$(1-2)' \ a \oplus b = b \oplus a$$

$$(1-3)' \ (a \oplus b) \oplus c = a \oplus (b \oplus c)$$

$$(1-4)' \ a \oplus (b * c) = a \text{ absorption}$$

Theorem - Let $\langle L, \leq \rangle$ be a lattice in which $*$ and $\oplus$ denote the operation of meet and join respectively for any $a, b \in L$

$$a \leq b \Leftrightarrow a * b = a \Leftrightarrow a \oplus b = b$$

Proof :- We shall first prove that
 $a \leq b \Leftrightarrow a * b = a$.

In order to do this.

Let us assume that $a \leq b$ since the relation $\leq$ is reflexive, we have $a \leq a$.

Therefore $a \leq a * b$.

But from the definition of $a * b$, we have $a * b \leq a$.

Hence, $a \leq b \Rightarrow a * b = a$.

assume that $a * b = a$, but it is only possible if $a \leq b$.

i.e. $a * b = a \Rightarrow a \leq b$.

combining these two results, we get the required equivalent.

It is possible to show that $a \leq b \Leftrightarrow a \oplus b = b$ in a similar manner.

Alternatively, from $a * b = a$ we have by absorption law-

$$\begin{aligned} b \oplus (a * b) &= b \oplus a = a \oplus b. \\ \text{but,} \quad &= b \oplus (a * b) = b \end{aligned}$$

Hence, $a \oplus b = b$ follows from $a * b = a$.
By repeating similar steps.

we can show that $a * b = a$ follows from $a \oplus b = b$ and.
Hence, these are equivalent.

Theorem - let $\langle L, \leq \rangle$ be a lattice for any $a, b, c \in L$ the following properties called isotonicity hold.

$$b \leq c \Rightarrow \begin{cases} a * b \leq a * c \\ a \oplus b \leq a \oplus c \end{cases}$$

Proof :- (i) firstly we prove that

$$b \leq c \Leftrightarrow b * c = b.$$

In order to do this

let us assume that $b \leq c$ since the relation $\leq$ is a reflexive we have $b \leq b$.

Therefore $b \leq b * c$

But from the def of $b * c$
we have $b * c \leq b$.

Hence, $b \leq c \Rightarrow b * c = b$.

Then to show that

$a * b \leq a * c$, we shall show that
 $(a * b) * (a * c) = a * b$

$$\begin{aligned}\text{So, } (a * b) * (a * c) &= (a * a) * (b * c) \\ &= (b * c) * a \\ &= a * b.\end{aligned}$$

we have that

$$b \leq c \Rightarrow b \oplus c = b.$$

In order to do this

let us assume that $b \leq c$

Since the relation $\leq$ is reflexive

we have, $b \leq b$.

Therefore $b \leq b \oplus c$

But from the def of $b \oplus c$

we have, $b \oplus c \leq b$.

Hence $b \leq c \Rightarrow b \oplus c = b$.

Then to show that.

$$a \oplus b \leq a \oplus c.$$

we shall show that

$$\begin{aligned}(a \oplus b) \oplus (a \oplus c) &= (a \oplus b). \\ \text{So, } (a \oplus b) \oplus (a \oplus c) &= (a \oplus b) \oplus (b \oplus c) \\ &= a \oplus (b \oplus c) \\ &= a \oplus b.\end{aligned}$$

Lattice as algebraic system.

Defⁿ - A lattice is an algebraic system
 $\langle L, *, \oplus \rangle$ with two binary operation
 $*$ and $\oplus$ on L which are both.

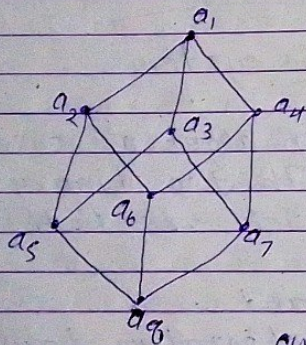
- 1) commutative
- 2) associative and
- 3) satisfy the absorption law.

* Sublattice :- let $\langle L, \oplus, * \rangle$ be a lattice
and let $S \subseteq L$ be a subset
of L . The algebra $\langle S, *, \oplus \rangle$ is a sublattice
of $\langle L, \oplus, * \rangle$ iff S is a closed under
both operations $*$ and $\oplus$.

for a lattice $\langle L, *, \oplus \rangle$ and for any two element $a, b \in L$, such that $a \leq b$ the closed interval $[a, b]$ consisting of all the element $x \in L$ such that $a \leq x \leq b$ is a sublattice of L .

Exam:- Let $\langle L, \leq \rangle$ be a lattice in which

$$L = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}$$



and S_1, S_2 and S_3 be a subset of given by $S_1 = \{a_1, a_2, a_4, a_6\}$
 $S_2 = \{a_3, a_5, a_6, a_7\}$
 and $S_3 = \{a_1, a_3, a_4, a_8\}$.

The diagram of $\langle L, \leq \rangle$ is given in

figure observe that $\langle S_1, \leq \rangle$ and $\langle S_2, \leq \rangle$ are sublattice of $\langle L, \leq \rangle$ but $\langle S_3, \leq \rangle$ is not a sublattice, because $a_2, a_4 \in S_3$ but $a_2 * a_4 = a_6 * a_3$ so, $\langle S_3, \leq \rangle$ is lattice.

Direct product of the lattice:-

Let $\langle L, *, \oplus \rangle$ and $\langle S, \cap, \vee \rangle$ be two lattice. The algebraic system $\langle L \times S, \cdot, + \rangle$ in which the binary operation $+$ and $\cdot$ on $L \times S$ are such that for any $\langle a_1, b_1 \rangle$ and $\langle a_2, b_2 \rangle$ in $L \times S$.

$$\langle a_1, b_1 \rangle \cdot \langle a_2, b_2 \rangle = \langle a_1 * a_2, b_1 \cap b_2 \rangle$$

$$\langle a_1, b_1 \rangle + \langle a_2, b_2 \rangle = \langle a_1 \oplus a_2, b_1 \vee b_2 \rangle.$$

is called the direct product of lattice $\langle L, *, \oplus \rangle$ and $\langle S, \cap, \vee \rangle$.

Lattice Homomorphism

Let $\langle L, *, \oplus \rangle$ and $\langle S, \vee, \wedge \rangle$ be two lattices. A mapping $g: L \rightarrow S$ is called a lattice homomorphism from the lattice $\langle L, *, \oplus \rangle$ to $\langle S, \vee, \wedge \rangle$ if for any $a, b \in L$,

$$g(a * b) = g(a) \wedge g(b)$$

$$g(a \oplus b) = g(a) \vee g(b)$$

Defⁿ - Let $\langle P, \leq \rangle$ and $\langle Q, \leq' \rangle$ be two partially ordered sets.

A mapping $f: P \rightarrow Q$ is said to be order preserving relative to the ordering $\leq$ in P and $\leq'$ in Q iff for any $a, b \in P$

Such that - $a \leq b$, $f(a) \leq' f(b)$ in Q .

If $\langle P, \leq \rangle$ and $\langle Q, \leq' \rangle$ are lattices and $g: P \rightarrow Q$ is a lattice homomorphism, then g is order preserving.

Some special lattice

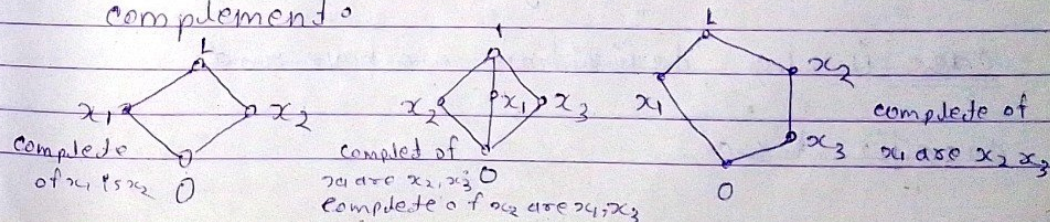
1] Complete lattice :- A lattice is called complete if each of its non-empty subsets has a least upper bound and greatest lower bound.

A lattice which has both elements 0 and 1 is called a bounded lattice.

2] Complement lattice :- In a bounded lattice $\langle L, *, \oplus, 0, 1 \rangle$ an element $b \in L$ is called a complement of an element $a \in L$ if

$$a * b = 0 \text{ and } a \oplus b = 1$$

3] Complemented lattice :- A lattice $\langle L, *, \oplus, 0, 1 \rangle$ is said to be a complemented lattice if every element of L has at least one complement.



Distributive Lattice :- A lattice $\langle L, *, \oplus \rangle$ is called a distributive lattice if for any $a, b, c \in L$

$$a * (b \oplus c) = (a * b) \oplus (a * c)$$

and $a \oplus (b * c) = (a \oplus b) * (a \oplus c)$

In other words in a distributive lattice the operation $*$ and $\oplus$ distributive over each other.

Theorem :- Every chain is distributive lattice

Proof :- We know by a definition that a chain is a linearly ordered set (or dually ~~orderly~~ ordered set) therefore every pair of element in this set passes a lub and a glb in the set as such a chain is a lattice.

Let $(L, \leq)$ be a chain and $a, b, c \in L$ consider the following possible case.

case (i) by definition we have and.

case (ii) $b \leq a$

case (i) $a \leq b$ or $a \leq c$
and

case (ii) $b \leq a$ and $c \leq a$

In case (i) by definition we have

$$a \oplus (b \oplus c) = a \quad \text{and} \quad (a * b) \oplus (a * c) = a$$

$$a * (b \oplus c) = (a \oplus b) \oplus (a \oplus c) \quad \text{--- (1)}$$

case (ii) by definition we have

$$a * (b \oplus c) = b \oplus c \quad \text{and}$$

$$(a * b) \oplus (a * c) = b \oplus c$$

$$\therefore a * (b \oplus c) = (a * b) \oplus (a * c) \quad \text{--- (2)}$$

Thus from (1) and (2) it follows that in both the cases and distributive law holds.

Similarly the second distributive law namely

$$a \oplus (b * c) = (a \oplus b) * (a \oplus c).$$

also holds by the principle of duality

Hence a chain is a distributive lattice.

Theorem - Every finite lattice is a bounded.

Proof:- Let $L = \{a_1, a_2, \dots, a_n\}$ be any finite lattice. we have to show that L having least and greatest element.

Now,

$$b_1 = a_1$$

$$b_2 = a_1 \wedge a_2$$

$$b_3 = a_3 \wedge b_2$$

$$\vdots$$

$$b_n = a_n \wedge b_{n-1}$$

Then $b_1, b_2, \dots, b_n$ all belongs to L and $b_n \leq a_i \quad i = 1, 2, 3, \dots, n$.

$\Rightarrow b_n$ is the least element of L
 $b_n = a_1 \wedge a_2 \wedge a_3 \dots \wedge a_n$

Similarly we can show that $a_1 \vee a_2 \vee a_3 \dots \vee a_n$ is the greatest element of L .

This is bounded lattice.

Boolean Algebra :-

A boolean algebra is a complemented distributive lattice.

A boolean algebra will generally denote by $\langle B, *, \oplus, 0, 1 \rangle$ in which $\langle B, *, \oplus \rangle$ is a lattice with two binary operations $*$ and $\oplus$ called the meet and join respectively.

Various boolean identities -

A boolean algebra $\langle B, *, \oplus, 0, 1 \rangle$ satisfies the following properties in which a, b, c denoted any element of the set B .

1] $\langle B, *, \oplus \rangle$ is a lattice in which the operation $*$ and $\oplus$ satisfy the following identities.

- (1-1) $a * a = a$ (1-1)' $a \oplus a = a$
- (1-2) $a * b = b * a$ (1-2)' $a \oplus b = b \oplus a$
- (1-3) $(a * b) * c = a * (b * c)$
- (1-4) $(a * (a \oplus b)) = a$
- (1-3)' $(a \oplus b) \in c = a \oplus (b \oplus c)$
- (1-4)' $a \oplus (a * b) = a$

2]. $\langle B, *, \oplus \rangle$ is a distributive lattice and satisfy these identities.

$$(D-1) a * (b \oplus c) = (a * b) \oplus (a * c)$$

$$(D-2) a \oplus (b * c) = (a \oplus b) * (a \oplus c)$$

$$(D-3) (a * b) \oplus (b * c) \oplus (c * a) = (a \oplus b) * (b \oplus c) * (c \oplus a)$$

$$(D-4) a * b = a * c \text{ and } a \oplus b = a \oplus c \Rightarrow b = c$$

3]. $\langle B, *, \oplus, 0, 1 \rangle$ is a bounded lattice in which for any $a \in B$ the following hold:

$$(B-1) 0 \leq a \leq 1$$

$$(B-2)' a \oplus 1 = 1$$

$$(B-2) a + 0 = a$$

$$(B-3)' a \oplus 1 = a$$

$$(B-3) a * 1 = a$$

4]. $\langle B, *, \oplus, 0, 1 \rangle$ is a uniquely complemented lattice in which the complement of any element $a \in B$ is denoted by $a' \in B$ and satisfies the following identities.

$$(C-1) a * a' = 0 \quad (C-1)' a \oplus a' = 1$$

$$(C-2) 0' = 1 \quad (C-2)' 1' = 0$$

$$(C-3) (a * b)' = a' \oplus b'$$

$$(C-3)' (a \oplus b)' = a' * b'$$

5]. There exists a partial ordering relation $\leq$ on B such that

$$(P-1) a * b = 1 \Leftrightarrow b \leq a \quad (P-1)' a \oplus b = 0 \Leftrightarrow b \leq a$$

$$(P-2) a \leq b \Leftrightarrow a * b = a \Leftrightarrow a \oplus b = b$$

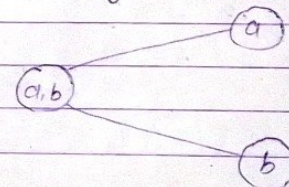
$$(P-3) a \leq b \Leftrightarrow a * b' = 0 \Leftrightarrow b' \leq a' \Leftrightarrow a' \oplus b = 1$$

Subalgebra :-

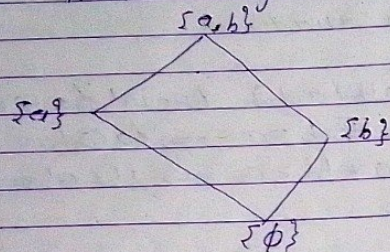
Let $\langle B, *, \oplus, 0, 1 \rangle$ be a boolean algebra and $S \subseteq B$. If S contains the element 0 and 1 and is closed under the operation $*$, $\oplus$ and $'$ then $\langle S, *, \oplus, 0, 1 \rangle$ is called a sub-boolean algebra.

Exa:- Draw the Hasse diagram $(P(A), \leq)$ for $A = \{a, b\}$ where $P(A)$ denotes the power set of A .

Soln- The diagram of the given poset is

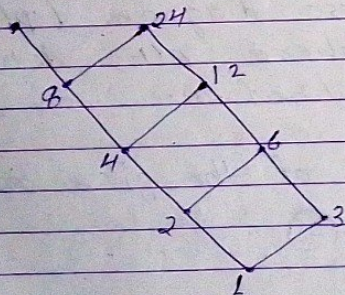


where $p(A) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$
and Hasse diagram:-



Exam:- Draw Hasse diagram of the poset ($D(24)$ divides), where $D(24)$ denote the set of all natural number divides 24 and the binary relation of the poset is divides.

Soln - We have,
 $D(24) = \{1, 2, 3, 4, 6, 8, 12, 24\}$.



Exa:- let L be the set of all factors of 12 and let ' $\mid$ ' be the divisibility relation on L .

show that $(L, \mid)$ is lattice.

Soln - By the definition on L we have

$$L = \{1, 2, 3, 4, 6, 12\}$$

let a, b be any two elements in L .
then $\sup \{a, b\}$ under divisibility relation is least positive integer c such that $a \mid c$ and $b \mid c$. Thus

$$a \vee b = \sup \{a, b\} = \text{lcm } \{a, b\}.$$

similarly, $\inf \{a, b\}$ is the greatest positive integer d such that $d \mid a$ and $d \mid b$.

$$\text{Hence } a \wedge b = \inf \{a, b\} = \text{gcd } \{a, b\}.$$

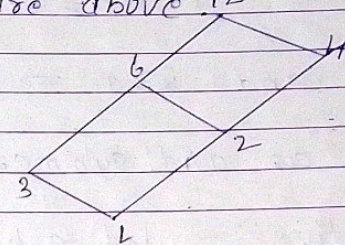
d.c.m.

x	1	2	3	4	6	12
1	1	2	3	4	6	12
2	2	2	6	4	6	12
3	3	6	3	12	6	12
4	4	4	12	4	12	12
6	6	6	6	12	6	12
12	12	12	12	12	12	12

g.c.d.

$\wedge$	1	2	3	4	6	12
1	1	1	1	1	1	1
2	1	2	1	2	2	2
3	1	1	3	1	3	3
4	1	2	1	4	2	4
6	1	2	3	2	6	6
12	1	2	3	4	6	12

from the above tables it is evident that for every $a, b \in L$ $\sup\{a, b\}$ and $\inf\{a, b\}$ exists in L . Thus $(L, \leq)$ is a lattice. This lattice is shown in the figure above.



Exam:- Show that if $(L, \leq)$ is a lattice then $(L, \geq)$ is also a lattice.

Soln:- let $(L, \leq)$ be a lattice and let $(L, \geq)$ be its dual, where the relation $\geq$ is defined as $a \geq b$ iff $b \leq a$.

We now show that $\geq$ is reflexive, anti-symmetric and transitive.

PO, $\geq$ is reflexive:- let $a \in L$ since $\leq$ is reflexive, we have $a \leq a \forall a \in L \Rightarrow a \geq a \forall a \in L \Rightarrow \geq$ is reflexive.

PO₂ $\geq$ is anti-symmetric :- let $a, b \in L$
be such that $a \geq b$

and $b \geq a$. Then
 $a \geq b$ and $b \geq a \Rightarrow b \leq a$ and $a \leq b$.
 $\Rightarrow a = b$.

Thus, $a \geq b$ and $b \geq a \Rightarrow a = b$

Hence $\geq$ is an anti-symmetric.

PO₃ is transitive :- let $a, b, c \in L$ be
such that

$a \geq b$ and $b \geq c$, then
 $a \geq b$ and $b \geq c \Rightarrow b \leq a$ and $c \leq b$
 $\Rightarrow c \leq b$ and $b \leq a$
 $\Rightarrow a \geq c$
 $\Rightarrow a \geq c$

Thus, $a \geq b$ and $b \geq c \Rightarrow a \geq c$

Hence, $\geq$ is transitive

Therefore, $\geq$ is a partial order relation
and L and so $(L, \geq)$ is a poset.

let $a, b \in L$, Then since $(L, \leq)$ is a
lattice.

2011 2022
Ques Show that in a lattice if $a \leq b$ and $c \leq d$
then $a * c \leq b * d$.

Soln - $a \leq b$ and $c \leq d$.
then $a * c \leq b * d$

$\Rightarrow a \leq b$ and $c \leq d \Rightarrow a * c \leq b * d$.

$a * c \leq a$ and $a * c \leq c$

and $a \leq b$ $c \leq d$.

Thus $a * c \leq b$

Thus $a * c \leq d$

$\Rightarrow (a * c) \leq b$ and $a * c \leq d \Rightarrow a * c$ is lub. of
 $\{b, d\}$

$\Rightarrow (a * c) \leq \{b, d\}$

$\Rightarrow a * c \leq b * d$.

$a * c \leq b * d$

show that in a lattice if $a \leq b \leq c$, then
 $a \oplus b = b * c$

$\Rightarrow a \leq b \leq c$

$\Rightarrow a \vee b = b * c$

$\Rightarrow a \leq b \Rightarrow a \vee b = b$

$\Rightarrow b \leq c \Rightarrow b * c = b$

$a \vee b = b * c$

Ans

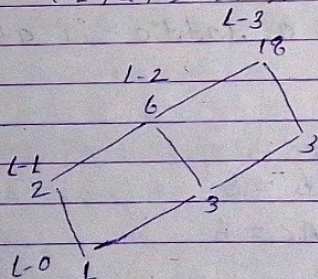
Hasse diagram - A graphical representation of a partial ordering relation in which all arrowheads are understood to be pointing upward is known as the Hasse diagram of the relation.

Exam :- The D_{18} be the set of all positive integer divisor of 18 and the partial order relation ' $<$ ' is defined by

$$a < b \Rightarrow a \mid b \quad \forall a, b \in D_{18}$$

Draw Hasse diagram -

$$D_{18} = \{1, 2, 3, 6, 9, 18\}$$



Dual of poset - Let R be a relation defined on set X . The converse of R denoted by $\bar{R}$ is a relation on X defined by

$$a \bar{R} b \Rightarrow b R a, a, b \in X$$

If R is a poset denoted by $\leq$ on X . Then $\bar{R}$ is a converse of R denoted by $\geq$.

Theorem :- If (X, R) is a poset then $(X, \bar{R})$ is also poset.

Proof :- We show that $\bar{R}$ is Reflexive, anti-symmetric and transitive.

(i) $\bar{R}$ is reflexive $\rightarrow$

Let $a \in X$, since R is reflexive.
 $a R a \quad \forall a \in X$
 $\Rightarrow a \bar{R} a \quad \forall a \in X$
 $\Rightarrow \bar{R}$ is reflexive.

② $\bar{R}$ is anti-symmetric $\rightarrow$

Let $a \bar{R} b$ and $b \bar{R} a$

Now $a \bar{R} b \Rightarrow b R a$

$b \bar{R} a \Rightarrow a R b$

$a R b$ and $b R a \Rightarrow a = b$

$a \bar{R} b$ and $b \bar{R} a \Rightarrow a = b$

Hence, $\bar{R}$ is anti-symmetric.

③ $\bar{R}$ is transitive

$a \bar{R} b$ and $b \bar{R} c \Rightarrow b R a$ and $c R b$

$\Rightarrow c R a$

$= a \bar{R} b$

$= \bar{R}$ is transitive

Therefore $\bar{R}$ is poset.

Ex (6) Let S be any non-empty set and $P(S)$ be its power set. $(P(S), \subseteq)$ is a lattice.

$\rightarrow$ Let S be any non-empty set and $P(S)$ be its power set.

Then we have to show that $(P(S), \subseteq)$ is a lattice.

But firstly we show that $(P(S), \subseteq)$ is poset

Define on relation R by.

$A R B \Leftrightarrow A \subseteq B \quad \forall A, B \in P(S)$

(i) R is reflexive:-

$\because A \subseteq A \Rightarrow \forall A \in P(S)$

$\Rightarrow A R A \quad \forall A \in P(S)$

(2). Thus R is reflexive.

(3). R is anti-symmetric:-

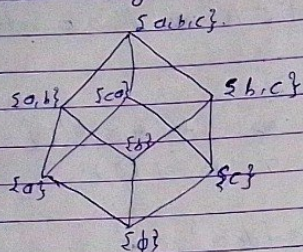
$\because A \subseteq B$ and $B \subseteq A \Rightarrow A \subseteq C$

$\Rightarrow A R C$

Now we show that $(P(S), \subseteq)$ is lattice.

Exam:- Consider $S = \{a, b, c\}$
 $P(S) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$

The hasse diagram of $(P(S), \subseteq)$ is



$$\inf \{A, B\} = A \cap B$$

$$\sup \{A, B\} = A \cup B$$

$\inf$	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{b, c\}$	$\{a, c\}$	$\{a, b, c\}$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{a\}$	$\emptyset$	$\{a\}$	$\emptyset$	$\emptyset$	$\{a\}$	$\emptyset$	$\{a\}$	$\{a\}$
$\{b\}$	$\emptyset$	$\emptyset$	$\{b\}$	$\emptyset$	$\{b\}$	$\{b\}$	$\emptyset$	$\{b\}$
$\{c\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\{c\}$	$\emptyset$	$\{c\}$	$\{c\}$	$\{c\}$
$\{a, b\}$	$\emptyset$	$\{a\}$	$\{b\}$	$\emptyset$	$\{a, b\}$	$\{b\}$	$\{a\}$	$\{a, b\}$
$\{b, c\}$	$\emptyset$	$\emptyset$	$\{b\}$	$\{c\}$	$\emptyset$	$\{b, c\}$	$\{c\}$	$\{b, c\}$
$\{a, c\}$	$\emptyset$	$\{a\}$	$\emptyset$	$\{c\}$	$\{a\}$	$\{c\}$	$\{a, c\}$	$\{a, c\}$
$\{a, b, c\}$	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{b, c\}$	$\{a, c\}$	$\{a, b, c\}$

(V) Sup $\emptyset$ $\{a\}$ $\{b\}$ $\{c\}$ $\{a, b\}$ $\{b, c\}$ $\{a, c\}$
 $\emptyset$ $\emptyset$ $\{a\}$ $\{b\}$ $\{c\}$ $\{a, b\}$ $\{b, c\}$ $\{a, c\}$
 $\{a\}$ $\{a\}$ $\{a\}$ $\{a, b\}$ $\{a, c\}$ $\{a, b\}$
 $\{b\}$ $\{b\}$ $\{a, b\}$ $\{b\}$ $\{b\}$ $\{a, b\}$
 $\{c\}$ $\{c\}$ $\{a, c\}$ $\{b, c\}$ $\{b, c\}$ $\{a, b, c\}$
 $\{a, b\}$ $\{a, b\}$ $\{a, b\}$ $\{a, b\}$ $\{a, b\}$ $\{a, b\}$
 $\{b, c\}$ $\{b, c\}$ $\{a, b, c\}$ $\{b, c\}$ $\{b, c\}$ $\{a, b, c\}$
 $\{a, c\}$ $\{a, c\}$ $\{a, c\}$ $\{a, c\}$ $\{a, b, c\}$ $\{a, b, c\}$
 $\{a, b, c\}$ $\{a, b, c\}$ $\{a, b, c\}$ $\{a, b, c\}$ $\{a, b, c\}$ $\{a, b, c\}$

from the above that -
every two etc of $P(S)$ has sup and inf
in $P(S)$ and hence $(P(S), \subseteq)$ is a lattice.

Ques - let S be a non-empty set and $P(S)$ be
its power set. Then algebraic system
 $\langle P(S), \cup \rangle$ and $\langle P(S), \cap \rangle$ are monoids with
the identities $\emptyset$ and S respectively.

Soln - let $S = \{a, b, c\}$ and then
 $P(S) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$
and binary operation " $\cup$ "

(1) Closure :- let us consider
 $\{a\} \cup \{b\} = \{a, b\} \in P(S) \forall \{a, b\} \in P(S)$
Hence $P(S)$ is closed with respect to $\cup$.

(2). Associative :- let $\{a\}, \{b\}, \{c\} \in P(S)$
 $\{(\{a\} \cup \{b\}) \cup \{c\} = \{a\} \cup \{b\} \cup \{c\}$

$$\begin{aligned} (\{a\} \cup \{b\}) \cup \{c\} &= \{a, b\} \cup \{c\} \\ &= \{a, b, c\} \\ &= \{a\} \cup \{b, c\} \\ &= \{a\} \cup (\{b\} \cup \{c\}) \end{aligned}$$

Hence, Element of $P(S)$ are associative with respect to $\cup$

(3). Identity :- $\forall \{a\} \in P(S) \exists \{\emptyset\} \in P(S)$
 such that, $\{a\} \cup \{\emptyset\} = \{a\}$ and
 $\{\emptyset\} \cup \{a\} = \{a\}$.

So there exist identity element in $P(S)$ w.r. to $\cup$.

For Binary operation $\cap$

(1) Closure - let us consider
 $\{a\} \cap \{b\} = \emptyset \in P(S)$
 $\forall \{a\}, \{b\} \in P(S)$

Hence, $P(S)$ is closure w.r. to $\cap$

(2) Associative :- let $\{a\}, \{b\}, \{c\} \in P(S)$

$$\Rightarrow (\{a\} \cap \{b\}) \cap \{c\} = \{a\} \cap (\{b\} \cap \{c\})$$

$$\begin{aligned} \text{L.H.S} - \{a\} \cap \{b\} \cap \{c\} &= \{\emptyset\} \cap \{c\} \\ &= \{\emptyset\} \end{aligned}$$

$$\begin{aligned} \text{R.H.S} - \{a\} \cap \{b\} \cap \{c\} &= \{a\} \cap \{\emptyset\} \\ &= \emptyset \end{aligned}$$

$$\text{L.H.S} = \text{R.H.S.}$$

Theorem :- for any commutative monoid $(M, *)$,
 set of idempotent elements of M
 form submonoid.

Proof :- let $(M, *)$ be a commutative monoid then it has the following from property.

(1). closure property $a, b \in M \Rightarrow a * b \in M$.

(2). Associative property $[a * (b * c)] = [(a * b) * c]$.

(3). Identity property $[\exists e_m \in M \text{ such that } a * e_m = a \forall a \in M]$

(4). Commutative property $[a * b = b * a]$.

Let S be the set of idempotent elements of M .

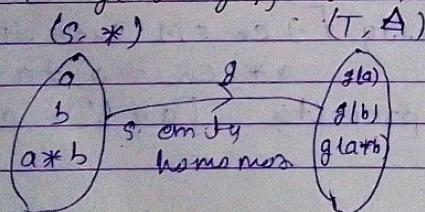
Then $S = \{x : x * x = x \forall x \in M\}$.

To prove that $(S, *)$ is a submonoid.

- 1) $a, b \in S \Rightarrow a * a = a$ and $b * b = b$.
 $\Rightarrow (a * a) * (b * b) = a * b$
 $\Rightarrow a * (a * b) * b = a * b$
 $\Rightarrow a * (a * b) * b = a * b$
 $\Rightarrow (a * b) * (a * b) = a * b$
 $\Rightarrow a * b$ is an idempotent element.

- 2) Since $em * em = em$
 $\therefore em$ is an idempotent element

Theorem - Let $(S, *)$ and (T, Δ) be two semigroups and g be a semigroup homomorphism from $(S, *)$ to (T, Δ) corresponding to the homomorphism g , $\exists$ a congruence relation R on $(S, *)$ define by xRy iff $g(x) = g(y) \forall x, y \in S$.



Define a relation R on $(S, *)$ by xRy iff $g(x) = g(y) \forall x, y \in S$.

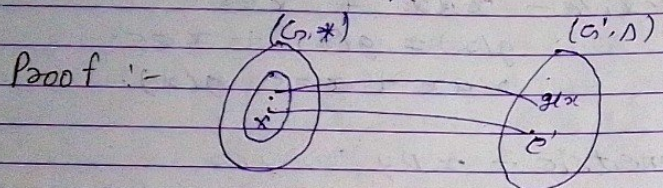
To prove that R is a congruence relation.

- ① R is reflexive - $xRx \forall x \in S$
 $g(x) = g(x) \forall x \in S$
 $xRx \forall x \in S$
- ② R is symmetric - $xRy \Rightarrow yRx$
 $xRy \Rightarrow g(x) = g(y)$
 $\Rightarrow g(y) = g(x)$
 $\Rightarrow yRx$
- ③ R is transitive - xRy and $yRz \Rightarrow xRz$
 xRy and $yRz \Rightarrow g(x) = g(y)$
 $\Rightarrow g(y) = g(z)$
 $g(x) = g(z)$
 $\Rightarrow xRz$

hence R is congruence relation.

Basic Homomorphism Theorem.

Statement :- Let g be a homomorphism of grp $(G, \cdot)$ onto (G', Δ) with kernel K then $\frac{G}{K} \cong G'$



$$\text{Let } K = \text{Ker } g = \{x \mid g(x) = e'\}$$

$$G' = \{g(x) \mid x \in G\}$$

$$\frac{G}{K} = \{Kx \mid x \in G\}$$

Define a mapping $\psi: \frac{G}{K} \rightarrow G'$ by

$$\psi(Kx) = g(x)$$

(1) ψ is one-one -

$$[\psi(Kx_1) = \psi(Kx_2) \Rightarrow Kx_1 = Kx_2]$$

$$\begin{aligned} \psi(Kx_1) = \psi(Kx_2) &\Rightarrow g(x_1) = g(x_2) \\ &\Rightarrow g(x_1) \Delta [g(x_2)]^{-1} = e' \end{aligned}$$

$$\Rightarrow g(x_1) \Delta g(x)^{-1} = e'$$

$$\Rightarrow g(x_1 x_2) = e' \quad g \text{ is homom}$$

$$\Rightarrow x_1 * x_2 \in K$$

$$\Rightarrow Kx_1 = Kx_2$$

Thus ψ is one-one

(2) ψ is onto - Let $g(x) \in G'$
 $\therefore x \in G$ and $Kx \in \frac{G}{K}$
 s.t. $\psi(Kx) = g(x)$
 $\therefore \psi$ is onto

(3) ψ is Homomorphism :-

$$[\psi(Kx_1 * Kx_2) = \psi(Kx_1) \Delta \psi(Kx_2)]$$

$$\begin{aligned} \psi(Kx_1 * Kx_2) &= \psi(K(x_1 * x_2)) \\ &= g(x_1 * x_2) \\ &= g(x_1) \Delta g(x_2) \\ &= \psi(Kx_1) \Delta \psi(Kx_2) \end{aligned}$$

Thus ψ is homomorphism.

Ques If $H_1, H_2, \dots, H_m$ and $p \Rightarrow Q$ then
 prove that $H_1, H_2, \dots, H_m$ implies
 $p \Rightarrow Q$.

Proof :- It is given that :-
 $H_1 \wedge H_2 \wedge \dots \wedge H_m \wedge p \Rightarrow Q$.
 To prove that.

$$H_1 \wedge H_2 \wedge \dots \wedge H_m \Rightarrow p \Rightarrow Q$$

$$H_1 \wedge H_2 \wedge \dots \wedge H_m \wedge p \Rightarrow Q$$

$$\equiv \neg (H_1 \wedge H_2 \wedge \dots \wedge H_m \wedge p) \vee Q$$

$$[\because A \Rightarrow B \equiv \neg A \vee B]$$

$$\equiv (\neg (H_1 \wedge H_2 \wedge \dots \wedge H_m) \vee \neg p) \vee Q$$

$$\equiv \neg (H_1 \wedge H_2 \wedge \dots \wedge H_m) \vee (\neg p \vee Q)$$

$$\equiv \neg (H_1 \wedge H_2 \wedge \dots \wedge H_m) \vee \neg (p \rightarrow Q)$$

$$[\because p \rightarrow Q \equiv \neg p \vee Q]$$

$$\equiv H_1 \wedge H_2 \wedge \dots \wedge H_m \Rightarrow (p \rightarrow Q)$$

$$[\because \neg A \vee B = A \Rightarrow B]$$